
9

LINEAR PROGRAMMING: SOLUTION METHODS

The modern methods for solving a large-scale integer program require the *optimization* and *reoptimization* of a usually long sequence of LP relaxation problems, which in turn are often solved by a variety of simplex-based methods. This chapter reviews three simplex-based methods that are the building blocks for solving integer programs. *The simplex method* provides the foundation for *optimizing* a long sequence of LP relaxations. *The simplex method for upper bounded variables* is used for reducing the problem size by *implicitly* handling the upper and lower bounds on variables (or *single-variable constraints*, more generally). *The dual simplex method* is most effective for *reoptimizing* the current optimum, after additional constraints are added, without resolving the augmented LP problem from scratch. *The revised simplex method* produces the same sequence of bases as the simplex method, but depends on updating $\mathbf{B}^{-1}$ (m columns) rather than on the entire simplex tableau (n columns).

9.1 LINEAR PROGRAMS IN CANONICAL FORM

Recall the following LP problem in standard form:

$$\text{Maximize } z = \sum_j c_j x_j \quad (9.1)$$

$$\text{subject to } \sum_j a_{ij}x_j \leq b_i \quad (i = 1, 2, \dots, m) \quad (9.2)$$

$$x_j \geq 0 \quad (j = 1, 2, \dots, n) \quad (9.3)$$

where b_i ($i = 1, 2, \dots, m$) can be a positive or negative number.

Because the simplex-based methods work on systems of linear equations rather than inequalities, the standard LP problem must be first converted to a system of equations. This can be accomplished by adding a nonnegative *slack variable* s_i to each inequality constraint (9.2) and transferring all the variable terms in the objective function (9.1) to the left-hand side of an equation and leaving the constant term on the right-hand side:

$$\text{Maximize } z - \sum_j c_j x_j = 0 \quad (9.4)$$

$$\text{subject to } \sum_j a_{ij}x_j + s_i = b_i \quad (i = 1, 2, \dots, m) \quad (9.5)$$

$$x_j \geq 0 \quad (j = 1, 2, \dots, n) \quad (9.6)$$

$$s_i \geq 0 \quad (i = 1, 2, \dots, m) \quad (9.7)$$

Based on this new formulation, therefore, solving a linear program can be viewed as performing the following three tasks:

1. Find solutions to the augmented system of linear equations in (9.4) and (9.5).
2. Use the nonnegative conditions (9.6) and (9.7) to indicate and maintain the feasibility of a solution.
3. Maximize the objective function, which is rewritten as equation (9.4).

Note that (9.4) can also be viewed as a combination of two parts:

$$\text{maximize } z$$

$$\text{subject to } z - \sum_j c_j x_j = 0$$

Also note that the augmented system of equations (9.4)–(9.5) has a particular form of coefficient matrix called *canonical form*. In this form, a solution can be read immediately from the right-hand side of all equations: all $x_j = 0$, all $s_i = b_i$, and $z = 0$. Clearly, a *feasible* solution is readily available if *all* $b_i \geq 0$. Moreover, this LP canonical form, after detaching the coefficients from the variables, appears to be the so-called *simplex tableau*.

If there exists any $b_i < 0$, the system has an *infeasible solution* because it violates at least one of (9.6) and (9.7). In this case therefore, to obtain a starting basic *feasible* solution, a Phase I problem must be constructed and solved. The details of this procedure will be given in Section 9.3.

9.2 BASIC FEASIBLE SOLUTIONS AND REDUCED COSTS

9.2.1 Basic Feasible Solution

Definition 9.1 Given that a system $\mathbf{Ax} = \mathbf{b}$, where the number of solutions are infinite, and $\text{rank}(\mathbf{A}) = m$ ($m < n$), a unique solution can be obtained by setting any $n - m$ variables to 0 and solving for the remaining system of m variables in m equations. Such a solution, if it exists, is called a *basic solution*. The variables that are set to 0 are called *nonbasic variables*, denoted by $\mathbf{x}_N$. The variables that are solved are called *basic variables*, denoted by $\mathbf{x}_B$. A basic solution that contains all nonnegative values is called a *basic feasible solution*. A basic solution that contains any *negative* component is called a *basic infeasible solution*. The $m \times m$ coefficient matrix associated with a given set of basic variables is called a *basis*, or *basis matrix*, and is denoted as $\mathbf{B}$.

Let $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)^T$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)^T$, and $\mathbf{A} = (\mathbf{B}, \mathbf{N})$. Then, the LP can be expressed by the following partitioned form:

$$\begin{aligned} \text{Maximize} \quad & z = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \\ & \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0} \end{aligned}$$

Example 9.1 Consider the following system of two equations in four unknowns (or variables):

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 + x_4 &= 8 \end{aligned}$$

A basic solution to this system can be obtained by assigning 0 to *any* two variables and solving the remaining system of two equations in two variables. This system has a maximum of six basic solutions:

$$C_2^4 = \frac{4!}{2!(4-2)!} = 6$$

These six basic solutions are listed in Table 9.1. Note that basic solutions 3 and 4 are *infeasible* because one of their basic variables has a *negative* value while the remaining basic solutions are *feasible*.

For $\mathbf{x}_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$, the basis $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and for $\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the basis $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

TABLE 9.1 Basic Solutions in Example 9.1

	Basic Solution					
	1	2	3	4	5	6
Nonbasic variables $\mathbf{x}_N$	$x_1 = 0,$ $x_2 = 0$	$x_1 = 0,$ $x_3 = 0$	$x_1 = 0,$ $x_4 = 0$	$x_2 = 0,$ $x_3 = 0$	$x_2 = 0,$ $x_4 = 0$	$x_3 = 0,$ $x_4 = 0$
Basic variables $\mathbf{x}_B$	$x_3 = 6,$ $x_4 = 8$	$x_2 = 6,$ $x_4 = 2$	$x_2 = 8,$ $x_3 = -2$	$x_1 = 6,$ $x_4 = -4$	$x_1 = 4,$ $x_3 = 2$	$x_1 = 2,$ $x_2 = 4$

In general, the number of basic solutions possible in a system of m equations in n variables is calculated by

$$C_m^n = \frac{n!}{m!(n-m)!}$$

We now formally define a canonical system of linear equations.

Definition 9.2 A system of linear equations is said to be in a *canonical form* if each equation contains a *basic variable* whose coefficient is 1 in that equation and whose coefficient in all other equations is 0.

Therefore, in a canonical system, every equation contains only one basic variable in the current basis whose value equals to the right-hand-side constant, and the rest of the variables are nonbasic with a value of 0. Thus, a basic solution can be obtained by letting each basic variable equal to the right-hand side of its respective equation and setting the nonbasic variables equal to zero.

Example 9.2 Consider the following LP in *standard form*:

$$\begin{aligned}\text{Max } z &= 4x_1 + 3x_2 \\ x_1 + x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0\end{aligned}$$

After transferring the objective function and adding nonnegative slack variables s_1 and s_2 to equalize the inequality constraints, we obtain a *canonical system*:

$$\begin{aligned}\text{Max } z - 4x_1 - 3x_2 &= 0 \\ x_1 + x_2 + s_1 &= 6 \\ 2x_1 + x_2 + s_2 &= 8 \\ x_1, x_2 &\geq 0 \\ s_1, s_2 &\geq 0\end{aligned}$$

Let x_1 and x_2 be nonbasic variables, then the remaining variables s_1 and s_2 are basic. A basic solution is $x_1 = x_2 = 0$, $s_1 = 6$, $s_2 = 8$, and $z = 0$, which forms a solution vector (including z -value component) equal to the right-hand side of the equations. For simplicity, we detach the coefficients from the variables resulting in a simplex tableau given in Table 9.2.

Note that z can also be viewed as a basic variable for the *objective equation (row)*. Just like *the constraint equations (or rows)*, this objective row is updated during

TABLE 9.2 Simplex Tableau for Example 9.2

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	-4	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	2	1	0	1	8

elementary row operations embedded in a simplex pivot. We shall refer to this equation as the *objective row* or *row 0* of the simplex tableau. The constraint rows are rows 1 through m .

9.2.2 Adjacent Basic Feasible Solution

Given a basic feasible solution in Table 9.2, one can generate an *adjacent* basic feasible solution by exchanging only one nonbasic variable for a basic variable while keeping all other variables unchanged (of course, their values typically change). This can be accomplished by the following steps: (a) determine a current nonbasic variable to become basic, (b) determine a current basic variable to become nonbasic, and (c) perform necessary row operations for exchanging the two variables determined in (a) and (b) and updating the values of the basic variables and z .

Definition 9.3 A nonbasic variable is called an *entering variable* if it is selected to become basic in the next basis. Its associated coefficient column is called a *pivot column*. A basic variable is called a *leaving variable* if it is selected to become nonbasic in the next basis. Its associated coefficient row is called a *pivot row*. The element that intersects a pivot column and a pivot row is called a *pivot or pivot element*. A *pivoting operation* is a sequence of elementary row operations that makes the pivot element “1” and all other elements “0” in the pivot column. Two basic feasible solutions are said to be *adjacent* if the set of their basic variables differ by only one basic variable. Geometrically, these two basic feasible solutions will correspond to two extreme points except in the case of degeneracy, in which case two or more bases correspond to the same extreme point.

In Table 9.2, suppose we select x_1 as the entering variable. This will make either s_1 or s_2 leave the basis. If s_1 is to leave, the coefficient of x_1 must be “1” in row 1 and “0” in both rows 2 and 0. To achieve this, we apply the following row operations: multiply row 1 by -2 and add to row 2, and then multiply row 1 by 4 and add to row 0, resulting in Table 9.3. Because s_2 is negative, the basic solution $(6, 0, 0, -4)$ is *infeasible*. So, selecting s_1 to exit was an improper choice.

Suppose x_1 is still the entering variable. If we let s_2 , instead of s_1 , leave the basis, then the coefficient of x_1 must be “1” in row 2 and “0” in rows 1 and 0. After pivoting, Table 9.4 is generated. Because all RHS values are nonnegative, the basic solution $(4, 0, 2, 0)$ is feasible with $z = 16$.

Note that the different results in Tables 9.3 and 9.4 indicate that a certain choice of leaving variable may cause the next basic solution *infeasible*. The question is how to

TABLE 9.3 Updated Simplex Tableau After Pivot 1

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	0	1	4	0	24
x_1	0	1	1	1	0	6
s_2	0	0	-1	-2	1	-4

TABLE 9.4 Updated Simplex Tableau After Pivot 2

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	0	-1	0	2	16
s_1	0	0	0.5	1	-0.5	2
x_1	0	1	0.5	0	0.5	4

ensure that the next basic solution will remain *feasible* if the current basic solution is *feasible*. To achieve this, we must choose a leaving variable such that its ratio of the right-hand side to the corresponding positive component in the pivot column is minimal. That is, from Table 9.1, we calculate the minimum ratio,

$$\min\{6/1, 8/2\} = 4$$

Because the minimum ratio 4 corresponds to s_2 row, s_2 must be the leaving variable to ensure the next basic solution is *feasible*.

Because row 0 of Table 9.4 still contains a negative value, which implies the objective value can be increased further, another simplex iteration is needed. Choosing x_2 as the entering variable and s_1 as the leaving variable, we obtain Table 9.5, which yields a *nonnegative* objective row and an optimum solution is found.

9.2.3 Reduced Costs

Examining Tables 9.2–9.5, we see that each row of a simplex tableau represents a basic variable written in terms of nonbasic variables. In Table 9.5 for instance, $x_2 + 2s_1 + s_2 = 4$ or $x_2 = 4 - 2s_1 + s_2$. In effect, the dimension of the original solution space is reduced from $(n + m)$, the number of basic and nonbasic variables,

TABLE 9.5 Optimal Tableau After Pivot 3

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	0	0	2	1	20
x_2	0	0	1	2	-1	4
x_1	0	1	0	-1	1	2

to a *subspace* of dimension n equal to the number of nonbasic variables. Premultiplying $\mathbf{B}^{-1}$ on the equation $\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$, we obtain

$$\mathbf{I}\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$

or $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$. Substituting it into $z = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N$, we have

$$\begin{aligned} z &= (\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T)\mathbf{x}_N \\ &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} - \sum_j (z_j - c_j)x_j, \quad j \text{ nonbasic} \end{aligned}$$

Definition 9.4 The subspace that contains only the nonbasic variables is referred to as a *reduced space*. The components of the objective row in a reduced space are called *reduced costs*, denoted by $\bar{\mathbf{c}}$:

$$\bar{\mathbf{c}}^T = (\bar{\mathbf{c}}_B^T, \bar{\mathbf{c}}_N^T) = (\mathbf{0}^T, \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T)$$

Note that the cost vector associated with the set of *basic variables* is a *null vector* $\mathbf{0}$.

9.3 THE SIMPLEX METHOD

The simplex method is an iterative algorithm consisting of the following steps:

1. *Initialization*: Find an initial basic solution that is feasible.
2. *Iteration*: Find a basic solution that is better, adjacent, and feasible.
3. *Optimality test*: Test if the current solution is optimal. If not, repeat step 2.

First, we shall address the iteration step, the core of the simplex method.

9.3.1 Better and Feasible Solution

The *iteration step* is aimed at finding a new basic solution that is *better*, *feasible*, and *adjacent* than a given feasible basic solution. When no better solution can be found, then an optimum solution has to be obtained. This iteration contains three basic steps: (1) determining the entering variable, (2) determining the leaving variable, and (3) pivoting on the pivot element for exchange of variables and updating the data in the tableau. A new basic solution will be *better* if an *entering variable* is properly chosen. A new basic solution will be *feasible* if a *leaving variable* is properly chosen. A new basic solution will be *adjacent* to the current one if only one basic variable from the old basic solution is exchanged with the old basic solution, which can be accomplished by a *pivot operation*. Consider the given simplex tableau in Table 9.6.

TABLE 9.6 The Simplex Tableau Immediately Before Pivoting

Basic Variable	$\mathbf{x_B}$						$\mathbf{x_N}$				RHS Solution	
	z	x_{B_1}	...	x_{B_r}	...	x_{B_m}	...	x_j	...	x_k	...	
z	1	0	...	0	...	0	...	$\bar{c}_j$	...	$\bar{c}_k$	...	$\bar{b}_o$
x_{B_1}	0	1	...	0	...	0	...	$\bar{a}_{1j}$	...	$\bar{a}_{1k}$	...	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_r}	0	0	...	1	...	0	...	$\bar{a}_{rj}$	...	$\bar{a}_{rk}$	...	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	0	...	0	...	1	...	$\bar{a}_{mj}$	...	$\bar{a}_{mk}$	...	$\bar{b}_m$

The entering variable, denoted by x_k , is chosen among the current nonbasic variables, denoted by $x_j \in \mathbf{x_N}$ ($j = 1, 2, \dots, n$), such that it will improve the current objective value. This can be accomplished by selecting the x_k with the most (or any) negative *reduced cost*. Mathematically,

$$x_k = \{x_j \in \mathbf{x_N} : \min_j \bar{c}_j, \bar{c}_j < 0\} \quad (9.8)$$

The coefficients column k associated with the entering variable x_k is the *pivot column*. The leaving variable, denoted by x_{B_r} , is chosen among the current basic variables, denoted by $x_{B_i} \in \mathbf{x_B}$ ($i = 1, 2, \dots, m$), such that it has a minimum (positive) ratio θ defined by

$$\theta = \frac{\bar{b}_r}{\bar{a}_{rk}} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}}, \bar{a}_{ik} > 0 \right\} \quad (9.9)$$

The coefficient row r associated with the leaving variable x_{B_r} is the *pivot row*. The rationale for selecting the minimum ratio is justified below.

Consider the simplex tableau given in Table 9.6. While holding $(n-1)$ nonbasic variables fixed at zero and increasing the nonbasic variable x_k from zero to positive, we will have the following system of equations for the objective function and constraints:

$$z + \bar{c}_k x_k = \bar{b}_o \quad \text{or} \quad z = \bar{b}_o - \bar{c}_k x_k \quad (9.10)$$

$$\text{and} \quad x_{B_i} + \bar{a}_{ik} x_k = \bar{b}_i \quad \text{or} \quad x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k \quad (i = 1, 2, \dots, m) \quad (9.11)$$

Because we want a *new* solution to remain *feasible*, meaning that the new x_{B_i} must be ≥ 0 for all i ,

$$x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k \geq 0 \quad \text{for} \quad i = 1, 2, \dots, m \quad (9.12)$$

If $\bar{a}_{ik} < 0$, then x_{B_i} increases as x_k increases and so x_{B_i} continues to be nonnegative without bound. If $\bar{a}_{ik} = 0$, then there is no change in x_{B_i} as x_k increases. Clearly, if $\bar{a}_{ik} \leq 0$ for all $i = 1, 2, \dots, m$, then the problem has an *unbounded solution*. Moreover, if there exists *any nonpositive column* j , not necessarily the pivot column, with a negative component in row 0 (or $\bar{c}_j < 0$), then x_j can increase to infinity without making the new x_{B_i} negative.

If $\bar{a}_{ik} > 0$, then x_{B_i} decreases as x_k increases. To satisfy nonnegativity, x_k is increased until the first basic variable x_{B_i} drops to zero. Examining the system of inequalities in (9.12), it is clear that the first basic variable dropping to zero corresponds to the minimum of $\bar{b}_i/\bar{a}_{ik}$ for positive $\bar{a}_{ik}$. Mathematically, we can increase x_k until equal to the amount of θ determined by (9.9). From (9.10), the new objective value will be $(\bar{b}_0 - \bar{c}_k\theta)$.

9.3.2 Updating Simplex Tableau by Pivoting

Now we address how to find an *adjacent* basic feasible solution. Given the entering variable x_k (pivot column k) and the leaving variable x_{B_r} (pivot row r), the *pivot element* $\bar{a}_{rk}$ can be determined by the intersection of row r and column k . To update the simplex tableau in Table 9.6, the following *pivoting operation* is performed.

- 1. Divide row r by $\bar{a}_{rk}$.
- 2. For all $i \neq r$, update the i th row by adding to it $(-\bar{a}_{ik})$ times the new r th row.
- 3. Update row 0 by adding to it $\bar{c}_k$ times the new r th row.

After pivoting operation, we obtain Table 9.7. Note that the positions of x_{B_r} and x_k are exchanged. That is, x_{B_r} appears in the rows of basic variables and x_k in the columns of nonbasic variables.

TABLE 9.7 The Simplex Tableau After Pivoting

Basic Variable	$\mathbf{x_B}$						$\mathbf{x_N}$					RHS Solution
	z	x_{B_1}	...	x_{B_r}	...	x_{B_m}	...	x_j	...	x_k	...	
z	1	0	...	$\frac{\bar{b}_r}{\bar{a}_{rk}}$	...	0	...	$\bar{c}_j - \frac{\bar{a}_{rj}}{\bar{a}_{rk}} \bar{c}_k$	...	0	...	$\bar{b}_0 - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{c}_k$
x_{B_1}	0	1	...	$\frac{\bar{a}_{1k}}{\bar{a}_{rk}}$	...	0	...	$\bar{a}_{1j} - \frac{\bar{a}_{rj}}{\bar{a}_{rk}} \bar{a}_{1k}$	...	0	...	$\bar{b}_1 - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{a}_{1k}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_k	0	0	...	$\frac{1}{\bar{a}_{rk}}$	...	0	...	$\frac{\bar{a}_{rj}}{\bar{a}_{rk}}$	...	1	...	$\frac{\bar{b}_r}{\bar{a}_{rk}}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	0	...	$\frac{\bar{a}_{mk}}{\bar{a}_{rk}}$	...	1	...	$\bar{a}_{mj} - \frac{\bar{a}_{rj}}{\bar{a}_{rk}} \bar{a}_{mk}$		0	...	$\bar{b}_m - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{a}_{mk}$

9.3.3 Optimality Test

An optimum solution is found if there exists no adjacent basic feasible solution that can improve the objective value. In a maximization problem, the optimality condition is satisfied if $\bar{c}_j \geq 0$ for all $j = 1, 2, \dots, m$. If at optimality, there exists a nonbasic variable, say x_p , with $\bar{c}_p = 0$, then this variable can enter the basis to obtain an alternate optimum solution with the same objective value.

9.3.4 Initial Basic Feasible Solution

In the preceding section, we assume that an LP problem has all constraints in $\leq$ form and all $b_i \geq 0$. In this case, a basic feasible solution is naturally obtained after adding a nonnegative slack variables s_i to each constraint. However, if there is any $b_i < 0$ or any constraint in $\geq$ or $=$ form, then *artificial variables* are added to become *basic variables for a starting basis*. Unfortunately, this basic solution is infeasible to the original problem because of the presence of artificial variables with positive values. To obtain a *feasible* basic solution to the original problem, a phase I problem of the two-phase method is constructed to drive all artificial variables out of basis (and hence equal 0). The construction procedure is given below.

1. Convert each constraint so that the right-hand side is nonnegative. This requires that any constraint with a negative right-hand side be multiplied by -1 . The resultant constraint has one of the three forms: $\leq$, $=$, or $\geq$. If it is in $\leq$ form, then add a nonnegative *slack variable*; in $=$ form, add a nonnegative artificial variable; in $\geq$ form, subtract a nonnegative slack variable and add a nonnegative *artificial variable*.
2. Solve a phase I problem whose objective function is minimizing the sum of artificial variables subject to the same set of constraints. The sum of artificial variables is obtained by assigning a cost of 1 to each artificial variable and 0 to each of nonartificial variables.

Example 9.3 This example is extended by adding the following additional constraint to Example 9.2.

$$-2x_1 + x_2 \geq 2$$

Applying step 1, we obtain

$$\begin{aligned} -2x_1 + x_2 - s_3 + x^a &= 2 \\ s_3, x^a &\geq 0 \end{aligned}$$

Applying step 2, we minimize $z^a = x^a$,

$$\begin{aligned} \text{or } \text{maximize } -z^a &= -x^a \\ \text{or } \text{maximize } -z^a + x^a &= 0 \end{aligned}$$

Setting up a tableau format for phase I problem, we have the following tableau:

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	0	0	0	0	0	1	0
s_1	0	1	1	1	0	0	0	6
s_2	0	2	1	0	1	0	0	8
x^a	0	-2	1	0	0	-1	1	2

Note that the above tableau is not yet in *canonical form* because the coefficient of x^a in row 0 is nonzero. To zero it out, we multiply row x^a by -1 and add the resultant row to row 0, resulting in the following tableau. Now the artificial variable x^a becomes a *basic* variable to the *transformed* problem.

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	2	-1	0	0	1	0	-2
s_1	0	1	1	1	0	0	0	6
s_2	0	2	1	0	1	0	0	8
x^a	0	-2	1	0	0	-1	1	2

Let x_2 be the entering variable and x_a be the leaving variable. After pivoting, we have the following tableau.

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	0	0	0	0	0	1	0
s_1	0	3	0	1	0	1	-1	4
s_2	0	4	0	0	1	1	-1	6
x_2	0	-2	1	0	0	-1	1	2

Because the artificial variable is driven out of basis and hence has a value of 0, we obtain a basic *feasible* solution for the *original* problem: $x_1 = 0$ and $x_2 = 2$.

Once a starting basic feasible solution is obtained, we proceed to the phase II problem to find an optimum solution using the original objective function and the last tableau of the phase I problem. To begin with, we must drop the columns associated with all the artificial variables, drop the objective row of phase I, and replace it with the original objective row. We obtain the following tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-4	-3	0	0	0	0
s_1	0	3	0	1	0	1	4
s_2	0	4	0	0	1	1	6
x_2	0	-2	1	0	0	-1	2

Note that this tableau is not yet in canonical form because in row 0 the coefficient of the basic variable x_2 is nonzero. To obtain a canonical form, we zero it out by multiplying row x_2 by 3 and adding the resultant row to the objective row.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-10	0	0	0	-3	6
s_1	0	3	0	1	0	1	4
s_2	0	4	0	0	1	1	6
x_2	0	-2	1	0	0	-1	2

This tableau has a basic feasible solution (0, 2, 4, 6, 0) but it is not optimal because the objective row contains negative components. Letting the entering variable be x_1 and the leaving variable be s_1 , we have the following tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	10/3	0	1/3	58/3
x_1	0	1	0	1/3	0	1/3	4/3
s_2	0	0	0	-4/3	1	-1/3	2/3
x_2	0	0	1	2/3	0	-1/3	14/3

Since all the components in the objective row are nonnegative, an optimum solution is found: (4/3, 14/3, 0, 2/3, 0) with an objective value $z = 58/3$.

9.4 INTERPRETING THE SIMPLEX TABLEAU

9.4.1 Entire Simplex Tableau

Every simplex tableau provides information about the *current* basic feasible solution and its n *adjacent* basic feasible solutions. Geometrically, a basic feasible solution corresponds to an extreme point of the feasible region. Recall that a convex hull of $n + 1$ points is called a *simplex*, hence the name of the simplex method.

9.4.2 Rows of Simplex Tableau

Every *row* of a simplex tableau represents an *equation* with all variable terms on the left-hand side and a constant term on the right-hand side of the equality sign. Moreover, the coefficients of all but one basic variables are zero. Note that the objective function can also be expressed as an equation, $z - \sum \bar{c}_j x_j = 0$ (where x_j is

nonbasic) with a new variable z being treated as a basic variable. This objective row is also referred to as row 0 and the remaining rows are rows 1 through m . The coefficient $\bar{c}_j = z_j - c_j$ is referred to a *reduced cost* because it is a cost coefficient expressed in the *reduced space* of n nonbasic variables.

Note that each equation contains exactly one basic variable with coefficient equal to 1 and one or more nonbasic variables with coefficients of any values. Moreover, different equations have distinct basic variables.

9.4.3 Columns of Simplex Tableau

The right-hand side column of a simplex tableau contains the objective value and the m values of basic variables for the current basic feasible solution. Note that the values of all nonbasic variables are always 0 and do not appear in the tableau.

The left-hand side column associated with each *basic variable* always contains a unit column vector with a “1” corresponding to the basic variable and a “0” to each of the nonbasic variables, including row 0. The left-hand side column associated with each *nonbasic variable* provides information about the basic feasible solutions adjacent to the current one. The objective component of the left-hand side column predicts the *negative* rate of the change in objective function value if the corresponding nonbasic variable is increased by one unit. The remaining components of the same left-hand side column predict the amount of each resource to be consumed if a nonbasic variable is increased by one. In case of a negative component, the resource is added rather than consumed. Therefore, in the calculation of minimum ratio, the negative and zero components are excluded.

9.4.4 Pivot Column and Pivot Row

The negative of row-0 component in the pivot column represents the unit improvement in the objective value if the entering variable is increased by one unit. The ratio of a right-hand side to a positive component of the pivot column represents the maximum amount that the corresponding nonbasic variable can be increased without exceeding the resource availability on the right-hand side of the equation. To satisfy the limits of all resources, a minimum ratio must be used. Otherwise, the new solution will be infeasible, indicated by negative values on the right-hand side.

9.4.5 Predicting the New Objective Value Before Updating

Prior to updating a simplex tableau, the new objective value of the next tableau can be predicted by the following formula:

$$\begin{aligned}\text{New } z &= \text{current } z + \text{total improvement in the objective value} \\ &= \text{current } z + (\text{unit improvement})(\text{amount of improvement})\end{aligned}$$

$$\text{or, } \hat{b}_o = \bar{b}_o - \bar{c}_k \theta$$

where $\hat{b}_o$ denotes the new objective value.

9.5 GEOMETRIC INTERPRETATION OF THE SIMPLEX METHOD

9.5.1 Basic Feasible Solution Versus Extreme Point

Recall that in Chapter 8, a point $\mathbf{x}$ in a polyhedron P (feasible region) is called an extreme point of P if it cannot be represented as a strict ($0 < \alpha < 1$) convex combination of two distinct points in P . Here we will show that every extreme point corresponds one-for-one to a basic feasible solution in the absence of degeneracy. To illustrate this, we compare the extreme points in Figure 9.1 with the basic feasible solutions in Table 9.1 for Example 9.2. Figure 9.1 shows a feasible region (polyhedron) with four extreme points and two infeasible points outside the feasible region: $(6, 0)$ and $(0, 8)$.

Table 9.1 lists four *basic feasible solutions* and two *basic infeasible solutions*. Comparing Table 9.1 with Figure 9.1, we see that every basic feasible solution corresponds to an extreme point. Furthermore, every basic infeasible solution corresponds to a point *outside* the feasible region. The two-dimensional figure does not show the values of slack variables s_1 and s_2 , whose values can be obtained by substituting the values of x_1 and x_2 into the respective equations. For example, extreme point $(6, 0)$ has $s_1 = 0$ and $s_2 = -4$ and extreme point $(0, 8)$ has $s_1 = -2$ and $s_2 = 0$. Both points are outside the feasible region and correspond to basic infeasible solutions.

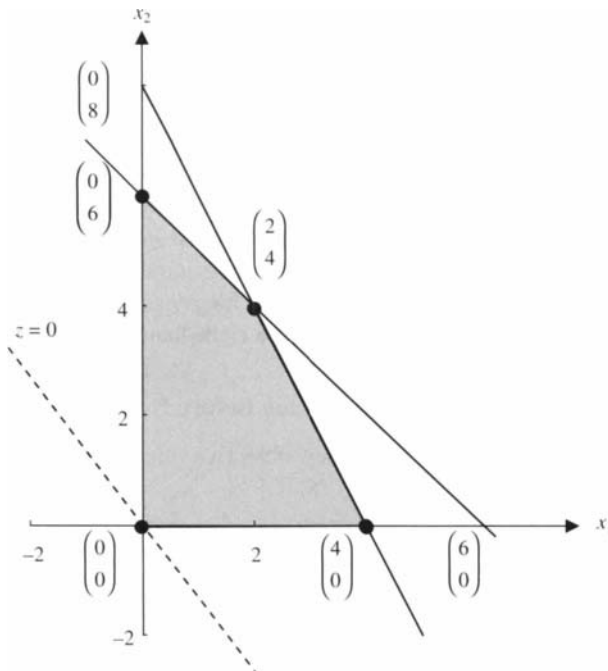


FIGURE 9.1 Six basic solutions to Example 9.2.

TABLE 9.8 Basic Solutions Associated with Figure 9.2

		Basic Solution								
	1	2	3	4	5	6	7	8	9	10
x_N	$x_1 = 0,$	$x_1 = 0,$	$x_1 = 0,$	$x_1 = 0,$	$x_2 = 0,$	$x_2 = 0,$	$x_2 = 0,$	$s_1 = 0,$	$s_1 = 0,$	$s_2 = 0,$
	$x_2 = 0$	$s_1 = 0$	$s_2 = 0$	$s_3 = 0$	$s_1 = 0$	$s_2 = 0$	$s_3 = 0$	$s_2 = 0$	$s_3 = 0$	$s_3 = 0$
x_B	$s_1 = 6,$	$x_2 = 6,$	$x_2 = 8,$	No solution	$x_1 = 6,$	$x_1 = 2,$	$x_1 = 4,$	$x_1 = 2,$	$x_1 = 4,$	$x_1 = 4,$
	$s_2 = 8,$	$s_2 = 2,$	$s_1 = -2,$		$s_2 = -4,$	$s_1 = 4,$	$s_1 = 2,$	$x_2 = 4,$	$x_2 = 2,$	$x_2 = 0,$
	$s_3 = 4$	$s_3 = 4$	$s_3 = 4$		$s_3 = -2$	$s_3 = -2$	$s_2 = 0$	$s_3 = 2$	$s_2 = -2$	$s_1 = 2$

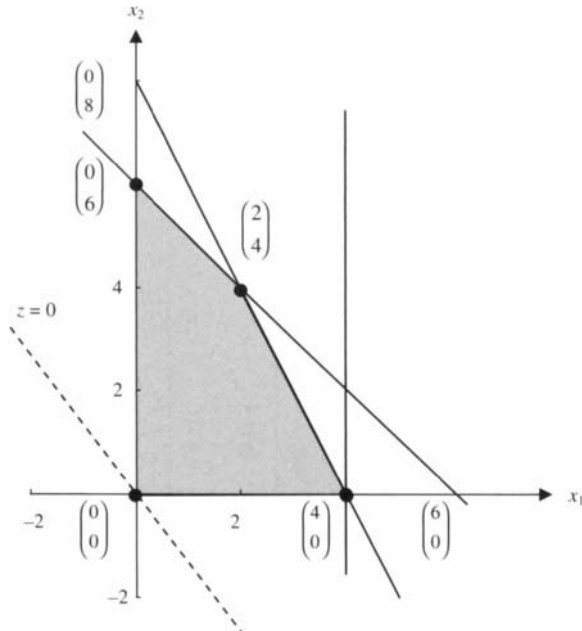


FIGURE 9.2 A degenerate solution created by $x_1 \leq 4$.

A degenerate solution has more than one basic feasible solution corresponding to an extreme point. To show this, we add the constraint $x_1 \leq 4$ to the problem in Example 9.2. Figure 9.2 shows the same feasible region with four extreme points, while Table 9.8 shows the five basic feasible solutions. Note that every basic feasible solution corresponds to an extreme point, except that the two *degenerate solutions* 7 and 10 correspond to the same extreme point $(4, 0)$. The only difference between the two is that they belong to different *bases*. In other words, they have two different sets of basic variables: the *zero-valued* variable x_2 is *basic* in solution 10, but *nonbasic* in solution 7.

9.5.2 Explanation of “Simplex Method” Nomenclature

Corresponding to *adjacent* basic feasible solutions, *adjacent* extreme points are hereby defined.

Definition 9.5 Two extreme points of a polyhedron X are said to be *adjacent* if they are joined by a line segment forming an edge of X .

Consider Figure 9.1. To the extreme point $(0, 0)$, for example, points $(4, 0)$ and $(0, 6)$ are its *adjacent extreme points*. To the extreme point $(4, 0)$, points $(2, 4)$ and $(0, 0)$ are its adjacent extreme points.

Recall that in Chapter 8, a specific class of the *bounded* polyhedron in E^n , formed by all convex combinations of $n + 1$ linear independent vectors, is called a *simplex*.

The *simplex method* searches the feasible region for an optimum extreme point by sequentially examining a subset of simplexes comprising the boundary of the polyhedron. Each simplex is formed by the convex combination of the current extreme point, a basic feasible solution, and n adjacent extreme points. Each simplex iteration geometrically moves from the current extreme point to an adjacent extreme point along an edge of one of these simplexes.

Take Figure 9.3a for an example. There are four extreme points and hence four simplexes. The four extreme points are denoted by $\mathbf{x}^1 = (0, 0)$, $\mathbf{x}^2 = (0, 6)$, $\mathbf{x}^3 = (2, 4)$, and $\mathbf{x}^4 = (4, 0)$. The four simplexes are $S_1 = \{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^4\}$, $S_2 = \{\mathbf{x}^2, \mathbf{x}^1, \mathbf{x}^3\}$, $S_3 = \{\mathbf{x}^3, \mathbf{x}^2, \mathbf{x}^4\}$, and $S_4 = \{\mathbf{x}^4, \mathbf{x}^3, \mathbf{x}^1\}$. Simplex S_1 is indicated by a shaded triangle in Figure 9.3a, and simplex S_2 by a shaded triangle in Figure 9.3b. Solving the LP problem in Example 9.2, the simplex method begins with S_1 and moves to S_4 , and then to S_3 when $\mathbf{x}^3$ is the optimum. The corresponding edges traveled are $[\mathbf{x}^1, \mathbf{x}^4]$ and $[\mathbf{x}^4, \mathbf{x}^3]$. An alternate sequence is $S_1 \rightarrow S_2 \rightarrow S_3$. Note that each simplex tableau contains the information about the current extreme point (basic feasible solution) and its adjacent extreme points. The pivot procedure decides whether to exchange the current extreme point for an adjacent extreme point and determines the coordinates (values) of the next extreme point.

9.5.3 Identifying an Extreme Ray in a Simplex Tableau

Recalling from Chapter 8, we know that an *unbounded* polyhedron P of an LP problem can be described in terms of *extreme points* and *extreme rays*. An extreme ray of P is defined as $\mathbf{x} = \mathbf{x}_0 + \lambda \mathbf{d}$, $\lambda \geq 0$, where $\mathbf{x}_0$ is the *root* or *vertex* of the extreme ray, $\mathbf{d}$ is the *extreme direction*, and λ is a nonnegative scalar, unbounded above. Note that $\mathbf{x}_0$ is an extreme point of P . In Chapter 8, we learned how to calculate algebraically an extreme direction. Here we will show how to identify an extreme ray and extreme direction from a given simplex tableau. First, we use a simple graphical example and then derive the algebraic relationship.

Example 9.4 (Extreme Ray) Consider the following LP problem:

$$\begin{aligned} \text{Maximize} \quad & z = 4x_1 + 3x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 4 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solving it by the graphical method, we obtain Figure 9.4 in which the extreme ray is expressed by

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \lambda = 0$$

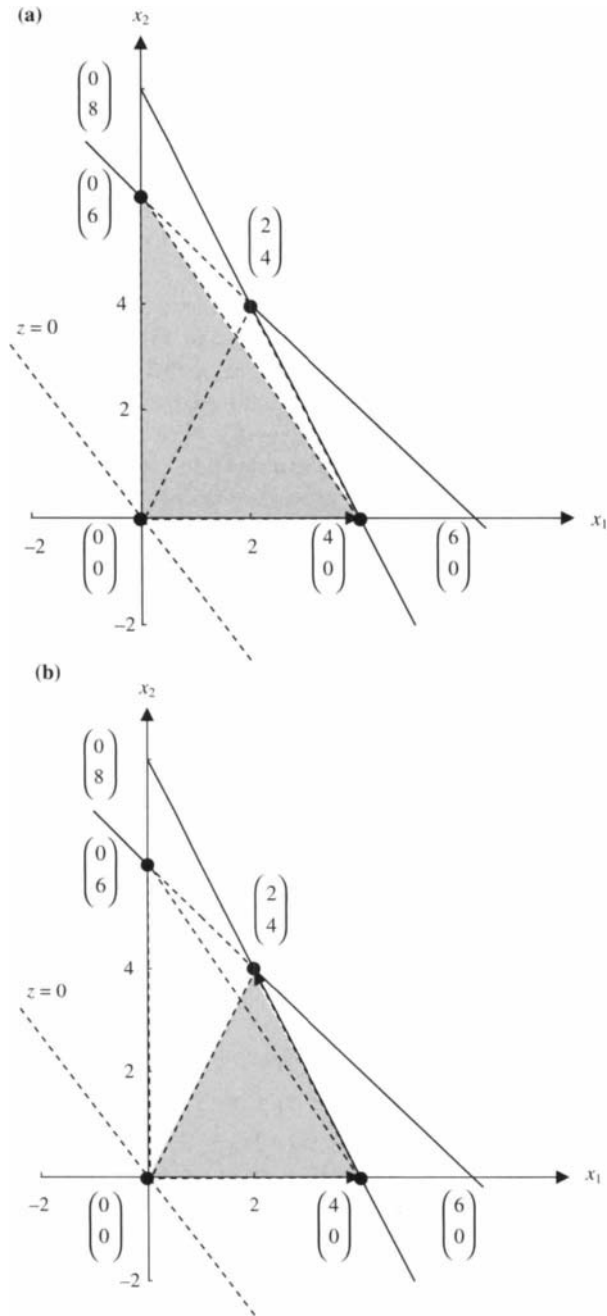


FIGURE 9.3 (a) A simplex associated with Example 9.2. (b) Another simplex associated with Example 9.2.

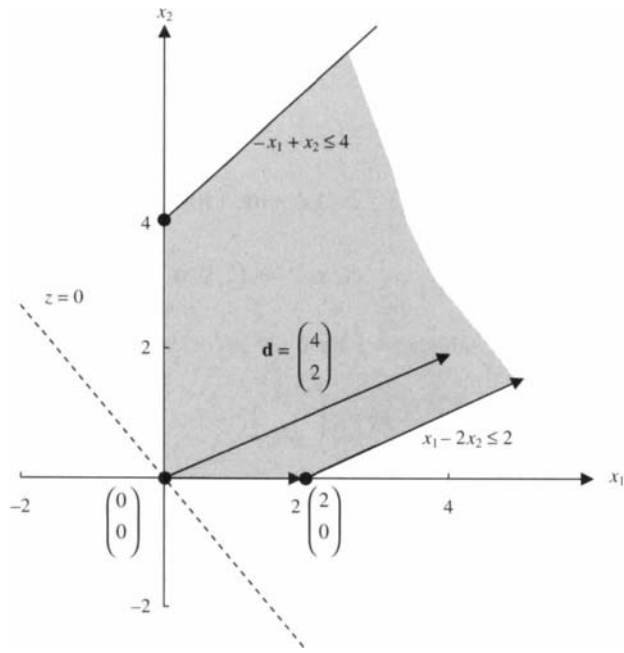


FIGURE 9.4 Feasible region for Example 9.4.

Alternatively, solving the problem by the simplex method, we obtain the following two simplex tableaus:

Basic Variable	z	x_1	x_2	x_3	x_4	RHS
z	1	-4	-3	0	0	0
x_3	0	-1	1	1	0	4
x_4	0	1	-2	0	1	2

Basic Variable	z	x_1	x_2	x_3	x_4	RHS
z	1	0	-11	0	4	8
x_3	0	0	-1	1	1	6
x_1	0	1	-2	0	1	2

The last simplex tableau indicates that the problem has an unbounded solution because x_2 column contains all nonpositive values. This condition implies that the variable x_2 can increase its value to ∞ without making the new RHS negative (or violating any constraints). This simplex tableau indicates that the

current basic feasible solution is

$$\mathbf{x}_B = (x_3, x_1)^T = (6, 2)^T$$

$$\text{and } \mathbf{x}_N = (x_2, x_4)^T = (0, 0)^T$$

Rearranging in order of variables, we have the following *root* of an extreme ray

$$\mathbf{x}_0 = (x_1, x_2, x_3, x_4)^T = (2, 0, 6, 0)^T$$

Also from this simplex tableau, we have the pivot column,

$$\bar{\mathbf{a}}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

Given that x_2 is the entering variable, the following condition must be satisfied to ensure that the next solution is feasible:

$$\begin{pmatrix} 2 \\ 0 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \end{pmatrix} x_2 \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $x_2 \geq 0$. The extreme direction of this ray is $\mathbf{d}^T = (2, 1, 1, 0)$, as can be seen in (9.16).

Now we show the general relationship algebraically. Recall that for a maximization problem, if we have a basic feasible solution with $\bar{c}_k < 0$ for some nonbasic variable x_k and $\bar{a}_{ik} \leq 0$ for all $i = 1, 2, \dots, m$, then the problem has an unbounded solution. This has been shown in Section 9.4.1 using the system of equations in (9.13), which is restated here:

$$x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k \quad (i = 1, 2, \dots, m)$$

$$\text{or in matrix form, } \mathbf{x}_B = \bar{\mathbf{b}} - \bar{\mathbf{a}}_k x_k$$

Because the coefficient of the entering variable x_k is 1, the vector of the next nonbasic variables must be

$$\mathbf{x}_N = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{e}_k$$

Putting $\mathbf{x}_B$ and $\mathbf{x}_N$ together, we obtain

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{b}} - \bar{\mathbf{a}}_k x_k \\ \mathbf{e}_k \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{b}} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} -\bar{\mathbf{a}}_k \\ \mathbf{e}_k \end{pmatrix} x_k \quad (9.13)$$

Comparing (9.13) with the definition of *extreme ray* given below,

$$\mathbf{x} = \mathbf{x}_0 + \lambda \mathbf{d}, \quad \lambda \geq 0 \quad (9.14)$$

with the *root*, the current extreme point,

$$\mathbf{x}_0 = \begin{pmatrix} \bar{\mathbf{b}} \\ \mathbf{0} \end{pmatrix} \quad (9.15)$$

the *extreme (ray) direction*,

$$\mathbf{d} = \begin{pmatrix} -\bar{\mathbf{a}}_k \\ \mathbf{e}_k \end{pmatrix} \quad (9.16)$$

and the unbounded step size

$$\lambda = x_k$$

9.6 THE SIMPLEX METHOD FOR UPPER BOUNDED VARIABLES

Quite often in practice, a variable has a lower bound other than 0 and has a finite upper bound other than infinity. Let l_j and u_j denote the lower and upper bounds of variable x_j , respectively. Then, we have the following lower bound and upper bound constraints as follows:

$$x_j \geq l_j \quad (9.17)$$

$$x_j \leq u_j \quad (9.18)$$

The lower bound constraint (9.17) can be easily handled by a variable substitution. Let a new variable $x'_j = x_j - l_j$ for any lower bound constraint. We can then obtain a

new problem containing new variables x'_j for all lower bound constraints, each with standard lower bound zero.

However, the upper bound constraint (9.18) cannot be handled similarly because despite substituting new variables $x''_j = u_j - x_j$, we still require upper bound constraints for the new variables. Nevertheless, the upper bound constraints can be handled *implicitly* by a modification of the ordinary simplex method without explicitly treating them as ordinary constraints. As a result, the number of constraints in an upper bounded linear program can be greatly reduced.

Now we are ready to describe the *simplex method for upper bounded variables*, also known as the *upper bound technique*. The basic concept is to allow any upper bound variable x_j to be nonbasic if $x_j = 0$ (as usual) or if $x_j = u_j$. To attain this, we use the following rules: For each upper bounded variable with $x_j = u_j$, we define a new variable $\bar{x}_j$ by the relationship $x_j + \bar{x}_j = u_j$ or $\bar{x}_j = u_j - x_j$. Note that if $x_j = 0$, then $\bar{x}_j = u_j$, whereas if $x_j = u_j$, then $\bar{x}_j = 0$. Whenever we want x_j to equal its upper bound u_j , we simply replace it with $u_j - \bar{x}_j$.

Suppose a basic feasible solution is available and we are solving a maximization problem. At each simplex iteration, we choose the entering variable x_k as in the ordinary simplex algorithm. There are three possible cases that limit the amount by which x_k can increase:

Case 1: x_k cannot exceed the minimum ratio θ as usual. Otherwise, it will cause one or more current basic variables to become negative.

Case 2: x_k cannot exceed the amount by which it will cause one or more current basic variables to exceed its upper bound. We shall denote this amount as θ' .

Case 3: x_k cannot exceed its upper bound u_k .

Any increase of x_k must be within these three limits (i.e., the minimum of θ , θ' , and u_k). The simplex algorithm for upper bounded variables for a maximization problem is as follows:

Step 0 (Initialization). Find a starting basic feasible solution as in the ordinary simplex method. Introduce a new variable $\bar{x}_j$ for each upper bound constraint, $x_j \leq u_j$, such that $x_j + \bar{x}_j = u_j$, where u_j is a constant.

Step 1 (Optimality Test). Check if the usual optimality condition is satisfied. If yes, an optimum solution is found; otherwise, go to next step.

Step 2 (Entering Variable). Select the entering variable x_k as in the ordinary simplex method.

Step 3 (Leaving Variable and Pivoting). Compute θ , θ' , and Δ as follows:

$$\theta = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}}, \bar{a}_{ik} > 0 \right\}$$

$$\theta' = \min_i \left\{ \frac{u_i - \bar{b}_i}{-\bar{a}_{ik}}, \bar{a}_{ik} < 0 \right\}$$

$$\Delta = \min\{\theta, \theta', u_k\}$$

There are three cases of Δ : (1) If $\Delta = \theta$, then determine the leaving variable x_{B_r} and perform the ordinary pivoting. (2) If $\Delta = \theta'$, then replace the leaving variable x_{B_r} with $u_{B_r} - \bar{x}_{B_r}$ in row r and the “label” for x_{B_r} with $\bar{x}_{B_r}$ and perform the ordinary pivoting. (3) If $\Delta = u_k$, then replace the entering variable x_k with $u_k - \bar{x}_k$ in each row of the tableau, and x_k with $\bar{x}_k$ in the “label” row. In any case, go to step 1 for an optimality test.

Example 9.5 (Bounded Variables) Consider the following LP problem

$$\begin{aligned} \text{Maximize} \quad & z = 4x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1 \geq 1 \\ & 1 \leq x_2 \leq 3 \end{aligned}$$

Assuming $x'_1 = x_1 - 1$, $x'_2 = x_2 - 1$, the problem can be transformed into

$$\begin{aligned} \text{Maximize} \quad & z = 4x'_1 + 3x'_2 + 7 \\ \text{subject to} \quad & x'_1 + x'_2 + s_1 = 4 \\ & 2x'_1 + x'_2 + s_2 = 5 \\ & x'_2 \leq 2 \\ & x'_1, x'_2 \geq 0 \end{aligned}$$

Let $x'_2 + \bar{x}'_2 = 2$. The starting basis consists of (s_1, s_2) with the initial tableau as follows:

Basic Variable	z	x'_1	x'_2	s_1	s_2	RHS
z	1	-4	-3	0	0	7
s_1	0	1	1	1	0	4
s_2	0	2	1	0	1	5

Clearly, the optimality condition is not satisfied. Select x'_1 as the entering variable. Then, we have

$$\theta = \min \left\{ \frac{4}{1}, \frac{5}{2} \right\} = 2.5$$

Note that θ' does not exist in this case since both $\bar{a}_{11}$ and $\bar{a}_{21}$ are nonnegative and that x'_1 has no upper bound. Thus, $\Delta = \theta = 2.5$, which makes s_2 the leaving variable. After pivoting, we obtain the following updated tableau.

Basic Variable	z	x'_1	x'_2	s_1	s_2	RHS
z	1	0	-1	0	2	17
s_1	0	0	0.5	1	-0.5	1.5
x'_1	0	1	0.5	0	0.5	2.5

The optimality condition still does not hold, so choose x'_2 to be the entering variable. Compute

$$\theta = \min \left\{ \frac{1.5}{0.5}, \frac{2.5}{0.5} \right\} = 3$$

Note that θ' does not exist. Since x'_2 has upper bound 2, $\Delta = \min\{\theta = 3, u'_2 = 2\} = 2$. Replacing x'_2 with $2 - \bar{x}'_2$, we obtain the following tableau:

Basic Variable	z	x'_1	$\bar{x}'_2$	s_1	s_2	RHS
z	1	0	1	0	2	19
s_1	0	0	-0.5	1	-0.5	0.5
x'_1	0	1	-0.5	0	0.5	1.5

Now the optimality condition is satisfied; hence, the optimal solution to the transformed problem is $x'_1 = 1.5$, $\bar{x}'_2 = 0$ (or $x'_2 = 2 - 0 = 2$), and $z = 19$. Transforming back to the original problem using the relations $x'_1 = x_1 - 1$ and $x'_2 = x_2 - 1$, we have an optimal solution to the original problem: $x_1 = 2.5$, $x_2 = 3$, and $z = 19$.

Handling lower bounded variables by substitution and upper bounded variables by this method greatly increase the efficiencies for solving LP problems. To

illustrate this, suppose we are solving an LP with 100 bounded variables with 10 other constraints. If we use the ordinary simplex method, the size of basis for each tableau would be $210 \times 210 (=44,100)$. If we use these two handling techniques, the size would be only 10×10 or 100. Moreover, solving an integer program by the branch-and-bound method (to be covered in Chapter 11) mainly contains two branches using lower and upper bounded variables. The savings in computation are evident.

9.7 THE DUAL SIMPLEX METHOD

There are three uses of the dual simplex method: (1) finding a new LP optimum *after* one or more constraints are added to the current LP optimum, (2) finding a new LP optimum after changing the right-hand side of constraints, and (3) solving an ordinary linear program.

For cases 1 and 2, the addition of constraints or change of the right-hand side may cause the current basic solution to become *infeasible*. In other words, the augmented simplex tableau may contain *negative* values on the right-hand side while the objective row remains *nonnegative* (dual feasible) in a maximization problem. These are the typical *starting conditions* for the dual simplex method. For case 3 where the objective row contains some negative values, we can augment a *big-M artificial constraint* to the original simplex tableau and perform row operations to obtain a canonical form for a starting basis. See note 9.7 for details.

For solving an integer program, the first use is the most important for efficient *reoptimization* because the dual simplex method is applied within the IP algorithms such as the branch-and-bound, cutting plane, and branch-and-cut. The dual simplex algorithm for a maximization problem is described below.

Step 0 (Initialization). Obtain a starting dual feasible solution. In the ordinary simplex tableau, this implies that *all* components of the objective row are *nonnegative*, or the updated values $\bar{c} \geq 0$. Initially, we construct a *basic* solution with only *slack variables* as basic variables (no artificial variables are ever needed). This may cause some right-hand side values to become negative. In the initial simplex tableau, $\bar{b}$ may be equal to b where some components are negative.

Step 1 (Optimality Test). Check if $\bar{b} \geq 0$. If yes, the current solution is optimal. Otherwise, go to next step.

Step 2 (Leaving Variable). Determine the leaving variable x_r by selecting a pivot row r with the most negative value on the right-hand side, that is, $\bar{b}_r = \min_i \{\bar{b}_i; \bar{b}_i < 0\}$.

Step 3 (Infeasibility Test). If $\bar{a}_{rj} \geq 0$ for all j , the given problem has no feasible solution. Otherwise, go to next step.

Step 4 (Entering Variable). Determine the entering variable by selecting the pivot column k based on the minimum ratio test:

$$\frac{\bar{c}_k}{-\bar{a}_{rk}} = \min_j \left\{ \frac{\bar{c}_j}{-\bar{a}_{rj}}, \bar{a}_{rj} < 0 \right\}$$

where $\bar{c}_j$ is the j th component of $\bar{\mathbf{c}}_N$. Note that ties in entering variable are broken arbitrarily and that if the entering variable rule cannot be applied (e.g., $\bar{a}_{rj} \leq 0$ for all $j = 1, \dots, n$), then the dual is unbounded and the primal is infeasible.

Step 5 (Pivoting). Update the current simplex tableau by pivoting on the pivot element $\bar{a}_{rk}$. Return to step 1.

Example 9.6 (Dual Simplex) Consider the following constraint that is added after an optimum solution is found for Example 9.2: $3x_1 + 2x_2 \leq 12$. We wanted to find a new optimum using the dual simplex method. Recall the current optimum tableau (Table 9.5) below:

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	0	0	2	1	20
x_2	0	0	1	2	-1	4
x_1	0	1	0	-1	1	2

Appending the additional constraint in equation form after introducing a slack variable s_3 , we obtain the following tableau:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	2	1	0	20
x_2	0	0	1	2	-1	0	4
x_1	0	1	0	-1	1	0	2
s_3	0	3	2	0	0	1	12

Note that this tableau does not have a canonical form and hence is not a simplex tableau. To obtain a canonical form, we add to s_3 row (-2) multiple of x_2 row and (-3) multiple of x_1 row.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	2	0	0	20
x_2	0	0	1	2	-1	0	4
x_1	0	1	0	-1	1	0	2
s_3	0	0	0	-1	-1	1	-2

Because $\bar{c} \geq 0$ and $\bar{b}$ has a negative component, we have a starting condition for the dual simplex method. Let s_3 be the leaving variable. Compute $\min\{2/1, 0/1\}$, choose s_2 as the entering variable, and perform pivoting.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	2	0	0	20
x_2	0	0	1	3	0	0	6
x_1	0	1	0	-2	0	0	0
s_2	0	0	0	1	1	1	2

Because all $\bar{b}_i \geq 0$, we obtain an optimum solution $(0, 6, 0, 2)$ with $z = 20$.

9.8 THE REVISED SIMPLEX METHOD

Recall the LP problem in partitioned form

$$\begin{aligned} \text{Maximize } z &= \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ \text{subject to } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N &= \mathbf{b} \\ \mathbf{x}_B, \mathbf{x}_N &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{x}_B$ and $\mathbf{x}_N$, respectively, denote vectors of basic and nonbasic variables; $\mathbf{c}_B$ and $\mathbf{c}_N$, respectively, are associated objective coefficients; and $\mathbf{B}$ and $\mathbf{N}$, respectively, are coefficient matrices associated with the constraints.

Multiplying the equality constraints by $\mathbf{B}^{-1}$, we obtain

$$\mathbf{I}\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$

Writing the objective function in terms of $\mathbf{x}_N$, we obtain

$$\text{Maximize } z = (-\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} + \mathbf{c}_N^T) \mathbf{x}_N + \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$$

Transferring the variable term to the left-hand side of the objective row and combining it with the constraints, we obtain the following “ordinary” simplex tableau:

z	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
1	$\mathbf{0}^T$	$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T$	$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$
0	$\mathbf{I}$	$\mathbf{B}^{-1} \mathbf{N}$	$\mathbf{B}^{-1} \mathbf{b}$

From the foregoing ordinary simplex tableau, we observe the following notes:

1. Once the set of basic variables $\mathbf{x}_B$ is specified, the corresponding simplex tableau can be calculated directly from the original data.
2. There is no need to update and store the coefficient matrix associated with the basic variables because that matrix is always identity $\mathbf{I}$, and the reduced cost associated with each basic variables is always zero.
3. All other entries of the ordinary simplex tableau are characterized by pre-multiplying the original data by $\mathbf{B}^{-1}$. Therefore, only $\mathbf{B}^{-1}$ is required to be updated.
4. The reduced cost vector $\bar{\mathbf{c}}_N^T$ associated with the nonbasic variables is calculated by $\bar{\mathbf{c}}_N^T = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T$, whose j th entry $\bar{c}_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j$, where column $\mathbf{a}_j \in \mathbf{N}$.
5. The dual solution $\mathbf{u}^T$ is updated by $\mathbf{u}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$, which requires to be computed only once for each simplex iteration.
6. The updated coefficient columns associated with nonbasic variables are $\mathbf{B}^{-1} \mathbf{N}$, whose j th column is $\bar{\mathbf{a}}_j = \mathbf{B}^{-1} \mathbf{a}_j$.
7. The primal solution is computed by $\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} = \bar{\mathbf{b}}$, $\mathbf{x}_N = \mathbf{0}$, and $z = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} = \mathbf{c}_B^T \bar{\mathbf{b}}$.

Based on the foregoing results, the ordinary simplex tableau can be further simplified to the *revised simplex tableau* as shown below.

$\mathbf{c}_B^T \mathbf{B}^{-1} = \mathbf{u}^T$	$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} = z$
$\mathbf{B}^{-1}$	$\mathbf{B}^{-1} \mathbf{b} = \bar{\mathbf{b}}$

Using the above revised simplex tableau, the revised simplex algorithm for the maximization problem may now be described as follows:

Step 0 (Initialization). Find an initial revised simplex tableau using slack variables (and/or artificial variables, if needed). In the presence of artificial variables, the two-phase method is applied as usual to obtain a starting basic feasible solution. Initially, $\mathbf{B} = \mathbf{B}^{-1} = \mathbf{I}$, $\mathbf{u}^T = \mathbf{c}_B^T \mathbf{B}^{-1} = \mathbf{c}_B^T$, $\mathbf{B}^{-1} \mathbf{b} = \mathbf{b}$, and $z = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} = \mathbf{c}_B^T \mathbf{b}$.

Step 1 (Pivot Column). For all nonbasic j and $\mathbf{a}_j \in \mathbf{N} = \mathbf{A} \setminus \mathbf{B}$, compute

$$\begin{aligned}\bar{\mathbf{a}}_j &= \mathbf{B}^{-1} \mathbf{a}_j \\ \bar{c}_j &= \mathbf{u}^T \bar{\mathbf{a}}_j - c_j\end{aligned}$$

Determine the entering variable x_k by

$$k = \{j : \min_j(\bar{c}_j : \bar{c}_j < 0)\}$$

and the pivot column

$$\begin{pmatrix} \bar{c}_k \\ \bar{\mathbf{a}}_k \end{pmatrix}$$

Step 2 (Optimality). A finite maximum solution has been found if $\bar{c}_k \geq 0$ and at least one entry of $\bar{\mathbf{a}}_k$ is positive. An unbounded maximum exists if $\bar{c}_k \geq 0$ and $\bar{\mathbf{a}}_k \leq 0$. Otherwise, go to step 3.

Step 3 (Pivot Row). Append pivot column k and determine the leaving variable x_{B_r} or pivot row r by

$$r = \left\{ i : \min_i \left(\frac{\bar{b}_i}{\bar{a}_{ik}}, \bar{a}_{ik} > 0 \right) \right\}$$

Go to step 4.

Step 4 (Pivoting). Update the revised simplex tableau by pivoting at $\bar{a}_{rk}$. Go to step 1.

The revised simplex method is actually the version of the simplex algorithm most implemented in software. It is also particularly useful in the branch-and-price algorithm for solving MIPs (Chapter 13). In the standard simplex, the most computation time would be due to updating every column in the simplex tableau with each iteration. Even though this assures the availability of all the data needed for the next pivot (pivot position not yet determined), we actually end up using only one column to make the decision on which variable will exit the basis. An example will illustrate the efficiency of this approach.

Example 9.7 Solve the following problem by the revised simplex method.

$$\text{Maximize} \quad 4x_1 + 3x_2 + x_3 + 7x_4 + 6x_5$$

$$\text{subject to} \quad x_1 + 2x_2 + 3x_3 + x_4 - 3x_5 \leq 9$$

$$2x_1 - x_2 + 2x_3 + 2x_4 + x_5 \leq 10$$

$$-3x_1 + 2x_2 + x_3 - x_4 + 2x_5 \leq 11$$

$$\mathbf{x} \geq \mathbf{0}$$

Adding slack variables, we have

$$\begin{aligned}
 \text{Maximize } z &= 4x_1 + 3x_2 + x_3 + 7x_4 + 6x_5 \\
 \text{subject to } x_1 + 2x_2 + 3x_3 + x_4 - 3x_5 + s_1 &= 9 \\
 2x_1 - x_2 + 2x_3 + 2x_4 + x_5 + s_2 &= 10 \\
 -3x_1 + 2x_2 + x_3 - x_4 + 2x_5 + s_3 &= 11 \\
 \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}
 \end{aligned}$$

Step 0 (Initialization)

$$\begin{aligned}
 \mathbf{x}_B &= (s_1, s_2, s_3)^T, \mathbf{x}_N = (x_1, x_2, x_3, x_4, x_5)^T, \mathbf{c}_B^T = (0, 0, 0), \mathbf{c}_N^T = (4, 3, 1, 7, 6), \\
 \mathbf{b} &= (9, 10, 11)^T, \mathbf{B} = \mathbf{B}^{-1} = \mathbf{I}, \mathbf{u}^T = \mathbf{c}_B^T \mathbf{B}^{-1} = (0, 0, 0), \\
 \bar{\mathbf{b}} &= \mathbf{B}^{-1} \mathbf{b} = (9, 10, 11)^T, z = 0
 \end{aligned}$$

Iteration 1

Compute $\bar{\mathbf{c}}_N^T = \mathbf{c}_N^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (-4, -3, -1, -7, -6)$. We select x_4 as the entering variable. Then the pivoting column 4 is calculated by

$$\mathbf{B}^{-1} \mathbf{a}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \bar{c}_4 = \mathbf{u}^T \bar{\mathbf{a}}_4 - c_4 = -7$$

Add the pivot column to the right of the revised simplex tableau, and s_2 becomes the leaving variable (why?).

z	0	0	0	0	-7
s_1	1	0	0	9	1
s_2	0	1	0	10	2
s_3	0	0	1	11	-1

After pivoting, the new tableau becomes

z	0	3.5	0	35	0
s_1	1	-0.5	0	4	0
x_4	0	0.5	0	5	1
s_3	0	0.5	1	16	0

Iteration 2

$\mathbf{u}^T = (0, 3.5, 0)$. Therefore, $\bar{\mathbf{c}}_N^T = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (3, -6.5, 6, 0, -2.5, 0, 3.5, 0)$. Then, x_2 is selected as the entering variable.

$$\mathbf{B}^{-1} \mathbf{a}_2 = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -0.5 \\ 1.5 \end{bmatrix}, \bar{c}_2 = \mathbf{u}^T \bar{\mathbf{a}}_2 - c_2 = -6.5$$

Add the pivot column to the right of the revised simplex tableau and s_1 becomes the leaving variable.

z	0	3.5	0	35	-6.5
s_1	1	-0.5	0	4	2.5
x_4	0	0.5	0	5	-0.5
s_3	0	0.5	1	16	1.5

After pivoting, the new tableau becomes

z	2.6	2.2	0	45.4	0
x_2	0.4	-0.2	0	1.6	1
x_4	0.2	0.4	0	5.8	0
s_3	-0.6	0.8	1	13.6	0

Iteration 3

$\mathbf{u}^T = (2.6, 2.2, 0)$. Therefore, $\bar{\mathbf{c}}_N^T = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (3, 0, 11.2, 0, -11.6, 2.6, 2.2, 0)$. Then x_5 is selected as the entering variable.

$$\mathbf{B}^{-1} \mathbf{a}_5 = \begin{bmatrix} 0.4 & -0.2 & 0 \\ 0.2 & 0.4 & 0 \\ -0.6 & 0.8 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1.4 \\ -0.2 \\ 4.6 \end{bmatrix}, \bar{c}_j = -11.6$$

Add the column to the right of the revised simplex tableau and s_3 becomes the leaving variable.

z	2.6	2.2	0	45.4	-11.6
x_2	0.4	-0.2	0	1.6	-1.4
x_4	0.2	0.4	0	5.8	-0.2
s_3	-0.6	0.8	1	13.6	4.6

After pivoting, the new tableau becomes

z	1.09	4.22	2.52	79.70	0
x_2	0.22	-0.04	0.30	5.74	1
x_4	0.17	0.43	0.04	6.39	0
x_5	-0.13	0.17	0.22	2.96	0

Iteration 4

$\mathbf{u}^T = (1.09, 4.22, 2.52)$. Therefore, $\bar{\mathbf{c}}_N^T = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (-2.04, 0, 13.22, 0, 0, 1.09, 4.22, 2.52)$. Then x_1 is selected as the entering variable.

$$\mathbf{B}^{-1} \mathbf{a}_1 = \begin{bmatrix} 0.22 & -0.04 & 0.30 \\ 0.17 & 0.43 & 0.04 \\ -0.13 & 0.17 & 0.22 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -0.61 \\ 0.913 \\ -0.43 \end{bmatrix}, \bar{c}_1 = -2.04$$

Add the column to the right of the revised simplex tableau and x_4 becomes the leaving variable.

z	1.09	4.22	2.52	79.70	-2.04
x_2	0.22	-0.4	0.30	5.74	-0.61
x_4	0.17	0.43	0.04	6.39	0.913
x_5	-0.13	0.17	0.22	2.96	-0.43

After pivoting, the new tableau becomes

z	1.48	5.19	2.62	94	0
x_2	0.33	0.33	0.33	10	0
x_1	0.19	0.48	0.05	7	1
x_5	-0.05	0.38	0.24	6	0

Iteration 5

$\mathbf{u}^T = (1.48, 5.19, 2.62)$. Therefore, $\bar{\mathbf{c}}_N^T = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (0, 0, 16.42, 2.23, 0, 1.48, 5.19, 2.62)$. Since all reduced costs are nonnegative, an optimum has been found and the algorithm terminates.

9.9 NOTES

Section 9.2

Many LP solvers provide several options for the user to select the entering variable. The options may include random selection, the first index, the most unit improvement, and the most total improvement.

Section 9.3

For finding the starting basic feasible solution, the big M method seems easier to comprehend and to compute manually, but it is not implementable in practice because we cannot give an appropriate value precisely for the big M . No matter what value you choose, for some problems it could be either too large or too small. In either case, it could cause great truncation and/or rounding errors that would mislead the solution. The two-phase procedure is the only method utilized in practice.

Theoretically, the degeneracy can cause a cycling problem for simplex iterations (see the example given by Beale, 1955). But in practice the cycling problem is highly unlikely. Although there are cycling prevention rules available (e.g., Bland's rule (Bland, 1977)) that guarantee finite convergence of the simplex algorithm, usually they are not implemented in commercial software because of substantial extra computational efforts.

Section 9.7

The starting dual feasible solution corresponds to the condition when a cutting constraint is introduced after an LP optimum is obtained. To reoptimize the augmented LP problem, the dual simplex method is much efficient than solving a new LP problem from scratch.

In case the dual simplex method is used as an independent algorithm for solving an LP problem from scratch and if the dual solution is infeasible, then make it feasible by adding the following redundant constraint (also called *artificial constraint*) obtained by summing over all nonbasic variables:

$$\sum_{j \in N} x_j \leq M$$

$$\text{or} \quad \sum_{j \in N} x_j + x_{n+1} = M$$

where M is a big value that can be set equal to the sum of finite upper bounds of all *nonbasic* variables and x_{n+1} is the associated slack variable. Perform the elementary row operations such that a *standard form* of the simplex tableau is obtained.

9.10 EXERCISES

9.1 Transform the following LP problem to canonical form.

$$\begin{array}{ll}\text{Minimize} & z = 3x_1 + 5x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 13 \\ & -x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0\end{array}$$

9.2 Consider the LP problem in Exercise 9.1 again. List all its extreme points and determine if each pair of points are adjacent.

9.3 Consider the following LP problem. Plot the feasible region and identify all the extreme points. Find the optimal solution by evaluating the objective function at each extreme point.

$$\begin{array}{ll}\text{Maximize} & z = 5x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 12 \\ & -2x_1 + x_2 \leq 7 \\ & x_1 - x_2 \geq 3 \\ & x_1 \leq 5 \\ & x_1, x_2 \geq 0\end{array}$$

9.4 The point $(1/3, 1/3, 1/3)$ is feasible for the following problem. Is it also a basic solution? Why or why not?

$$\begin{array}{ll}\text{Minimize} & z = 2x_1 + 3x_2 - 2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 1 \\ & x_1 - 2x_2 + 2x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

9.5 Consider the following LP problem. Verify that $\mathbf{x} = (0, 2.5, 0)^T$ is optimal and that the dual price of binding constraint 3 is 1.5.

$$\begin{array}{ll}\text{Minimize} & z = 2x_1 + 3x_2 + x_3 \\ & x_1 + x_2 + x_3 \leq 3 \\ & 2x_1 + 2x_3 \leq 3 \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & \mathbf{x} \geq \mathbf{0}\end{array}$$

9.6 Consider the following LP problem:

$$\begin{aligned} \text{Minimize} \quad & z = x_1 - 2x_2 - 2x_3 \\ \text{subject to} \quad & 2x_1 + 2x_2 + x_3 \leq 11 \\ & x_1 + x_2 - x_3 \geq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- (i) Rewrite the problem in canonical form.
 - (ii) Find an initial tableau using two-phase method (without solving the whole problem).
- 9.7** Go through the phase I of the following LP problem and stop at the phase I optimum. Is the original problem feasible? Why?

$$\begin{aligned} \text{Minimize} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \geq 29 \\ & x_1, x_2 \geq 0 \end{aligned}$$

9.8 Consider the following LP problem:

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 = 21 \\ & x_1 - x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- (i) Rewrite the problem in canonical form.
 - (ii) Solve the problem using two-phase method.
- 9.9** Solve the following LP problem using the primal simplex method.

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 - 3x_2 \leq 4 \\ & -x_1 + x_2 \leq 2 \\ & 3x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- 9.10** Suppose variable x_j leaves the basis at some iteration p . Is it possible that x_j enters the basis at the end of iteration $p + 1$? Why or why not?

- 9.11** Solve the following LP problem using the primal simplex method. Note that some variables are unrestricted in sign.

$$\begin{aligned}
 &\text{Maximize} && z = -3x_1 + x_2 - x_3 + x_4 \\
 &\text{subject to} && 2x_1 - x_2 - x_3 + x_4 \leq 8 \\
 &&& -2x_1 + 2x_2 + 2x_3 + 3x_4 \leq 10 \\
 &&& -x_1 + x_2 - 3x_3 + x_4 \leq 3 \\
 &&& x_2, x_4 \geq 0
 \end{aligned}$$

- 9.12** Consider LP problem in Exercise 9.11. Suppose at some iteration the basis consists of x_2 , s_1 , and s_3 , where s_1 and s_3 are the slack variables corresponding to the first and third constraints, respectively.
- (i) Decide the right-hand side values at this iteration without actually solving the problem.
 - (ii) Decide the objective value at this iteration.
- 9.13** Consider the following LP problem:

$$\begin{aligned}
 &\text{Minimize} && z = 13x_1 - 7x_2 + x_3 + 3x_4 - 5x_5 + x_6 \\
 &\text{subject to} && -x_1 + 2x_2 + x_3 + x_4 \leq 10 \\
 &&& 2x_1 - x_3 + x_4 + x_5 \leq 13 \\
 &&& x_2 + 3x_3 - x_4 - x_6 \leq 7 \\
 &&& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{aligned}$$

- (i) Solve the problem to optimum using the primal simplex method. At each iteration, apply the rule of “least index” when picking the entering variable. That is, among all the candidates of entering variable, always choose the one with least index.
 - (ii) At optimum, adjoin to the model the constraint $x_2 \leq 4$. Reoptimize the problem using the dual simplex method.
- 9.14** Solve the LP problem in Exercise 9.4 using the dual simplex method.
- 9.15** The shaded area of the Figure 9.5 shows the feasible region of an LP problem.
- (i) Show the inequalities of the constraints.
 - (ii) Start the primal simplex iteration with the basic feasible solution $(x_1, x_2) = (3, 0)$ and iterate until an optimal solution is obtained.

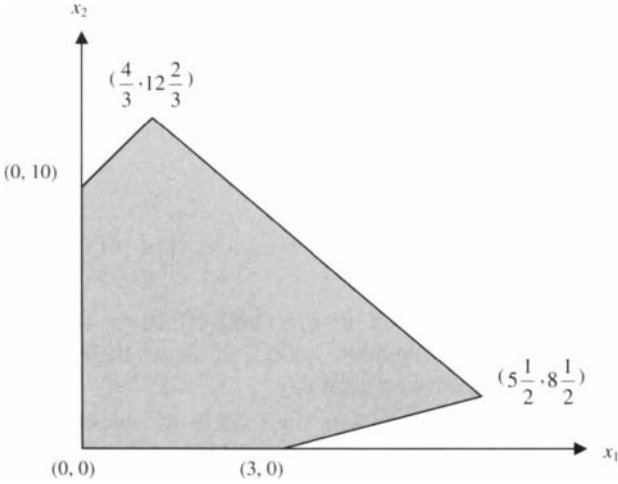


FIGURE 9.5 LP feasible region.

9.16 Consider the following LP problem:

Minimize $x_1 - 2x_2 + 2x_3 + x_4 - 5x_5 + 3x_6$
subject to $x_1 + x_2 + x_3 + x_5 \leq 7$
 $2x_2 - 2x_3 + 2x_5 + x_6 \leq 13$
 $x_2 \leq 3$
 $x_5 \leq 5$
 $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Solve it using the upper bound technique.

- 9.17** Consider the problem in Exercise 9.13. Now add two constraints: $x_1 \leq 5$, $x_2 \leq 8$.
- (i) Start from the optimal tableau you obtained in Exercise 9.15 and reoptimize by adding one constraint at a time.
 - (ii) What new extreme points are introduced into the problem, and which ones are gone?
 - (iii) Solve the whole problem from beginning by using upper bound technique.
- 9.18** Ben has three dogs, Uno, Dos, and Tres. Every day Ben feeds them with five types of food: beef, dog food, bread, bones, and chicken. Each type of food

TABLE 9.10 Food Prices

Food	Beef	Dog Food	Bread	Bones	Chicken
\$/lb	\$2.50	\$1.00	\$0.80	\$1.20	\$1.60

is bought in pounds. The price for each type of food is as follows (Table 9.10):

Ben’s wife wants him to find the most cost-effective plan for feeding dogs, subject to the dogs’ preferences. Table 9.11 shows the minimum amount of food consumed by each dog each day.

Despite this, Dos eats no less than 2.5 lb of chicken plus bread. Uno eats no less than 2.7 lb of meats (including dog food, chicken, and bones). The total amount of bread and beef fed to Tres cannot be less than 2.6 lb.

- (i) Formulate the problem as an LP.
- (ii) Solve the problem using LINGO®. Declare the variables and parameters by sets. Apply the domain defining functions as necessary. Show your output and interpret the solution.
- (iii) When solving the model, manipulate the “linear solver options” by selecting primal simplex, dual simplex, and barrier, respectively. Compare the number of steps it took for LINGO® to solve the problem under each option.

9.19 Solve the following LP problem using revised simplex method.

Maximize $-3x_1 + x_2 - x_3 + x_4$
subject to $2x_1 - x_2 - x_3 + x_4 \leq 8$
 $-2x_1 + 2x_2 + 2x_3 + 3x_4 \leq 10$
 $-x_1 + x_2 - 3x_3 + x_4 \leq 3$
 $\mathbf{x} \geq \mathbf{0}$

TABLE 9.11 Minimum Daily Food Consumption

	Beef	Dog Food	Bread	Bones	Chicken
Uno	0	0	0.5	1.7	1.9
Dos	0	1.5	0.3	0.9	0.1
Tres	1.5	0.9	0.8	0.6	0.2

9.20 Solve the following LP problem using revised simplex method.

$$\begin{array}{ll}\text{Minimize} & x_1 - 2x_2 - 2x_3 \\ \text{subject to} & 2x_1 + 2x_2 + x_3 \leq 11 \\ & x_1 + x_2 - x_3 \geq 4 \\ & x_1, x_2, x_3 \geq 0\end{array}$$