

PART II

REVIEW OF LINEAR PROGRAMMING AND NETWORK FLOWS

7

LINEAR PROGRAMMING— FUNDAMENTALS

This chapter reviews the basic linear algebra and linear programming theory essential to the understanding of the LP and IP solutions and methods to be discussed in the remainders of this book.

7.1 REVIEW OF BASIC LINEAR ALGEBRA

An n vector is a column or row array of n numbers. Throughout this text, we shall use a lowercase boldface letter to denote a column vector such as $\mathbf{b}$ or $\mathbf{d}$, and use the transpose of a column vector to denote a row vector such as $\mathbf{b}^T$ or $\mathbf{d}^T$, where superscript T stands for *transpose*.

7.1.1 Euclidean Space

Definition 7.1 An n -dimensional *Euclidean space*, denoted by E^n , is the collection of all vectors of dimension n having the following properties:

- *Addition of vectors*—for any two vectors $\mathbf{a}$ and $\mathbf{b}$ in E^n , vector $\mathbf{a} + \mathbf{b}$ is in E^n .
- *Scalar multiplication*—for any vector $\mathbf{a}$ and scalar k in E^n , vector $k\mathbf{a}$ is in E^n .

- *Vector multiplication*—any two vectors $\mathbf{a}$ and $\mathbf{b}$ in E^n can be multiplied. The result of this vector multiplication is a real number called *inner (dot) product of the two vectors*, defined by $\mathbf{a}^T \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.
- The *length (norm) of a vector* in the space, denoted by $\|\mathbf{a}\|$, is defined

$$\|\mathbf{a}\| = \sqrt{\sum_{i=1}^n a_i^2}$$

7.1.2 Linear and Convex Combinations

Definition 7.2 Given p vectors, $\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^p$ in E^n , and p scalars $\lambda_1, \lambda_2, \dots, \lambda_p$, the expression $\lambda_1 \mathbf{a}^1 + \lambda_2 \mathbf{a}^2 + \cdots + \lambda_p \mathbf{a}^p$ is called a *linear combination*. The scalars are real numbers that can be positive, negative, or zero. The linear combination becomes a *convex combination* when $\lambda_1 + \lambda_2 + \cdots + \lambda_p = 1$ and $0 \leq \lambda_1, \lambda_2, \dots, \lambda_p \leq 1$.

7.1.3 Linear Independence

Definition 7.3 A collection of vectors $\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^p$ of dimension n is called *linearly dependent* if there exist constants $\alpha_1, \alpha_2, \dots, \alpha_p$, with at least one $\alpha_j \neq 0$, such that their linear combination equals to an n -dimensional null vector, or

$$\sum_{j=1}^p \alpha_j \mathbf{a}^j = \mathbf{0}$$

and the set of vectors is said to be *linearly independent* if the only solution is $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$.

Note that in E^n , any set of $p > n$ vectors is always linearly dependent, but a set of $p \leq n$ vectors may or may not be linearly independent.

7.1.4 Rank of a Matrix

A matrix is a rectangular array of numbers. Thus, the size of a matrix is represented by the number of rows crossed with the number of columns, denoted by $m \times n$. In this text, we shall use an uppercase boldface letter to denote a matrix, for example, $\mathbf{A}$ or $\mathbf{B}$.

Definition 7.4 Let $\mathbf{A}$ be an $m \times n$ matrix. The *row rank* of the matrix is equal to the maximum number of linearly independent rows of $\mathbf{A}$. The *column rank* of $\mathbf{A}$ is the maximum number of linearly independent columns of $\mathbf{A}$. It can be shown that the row

rank of a matrix is always equal to its column rank, and hence the rank of $\mathbf{A}$, denoted by $\text{rank}(\mathbf{A})$, is equal to the number of linearly independent rows (or columns) of $\mathbf{A}$.

7.1.5 Basis

A collection of vectors $\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^p$ in E^n is said to *span* E^n if any vector in this space can be represented as a linear combination of this set of vectors. In other words, given any vector $\mathbf{b}$ in this space, we can always find scalars $\alpha_1, \alpha_2, \dots, \alpha_p$ such that

$$\mathbf{b} = \sum_{j=1}^p \alpha_j \mathbf{a}^j$$

Definition 7.5 A collection of n linearly independent row vectors forms a *basis* of E^n .

Thus, given a basis of E^n , say $\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^n$, any vector $\mathbf{b}$ in E^n is *uniquely* represented in terms of this basis. Note that a basis must be an $n \times n$ square matrix.

7.1.6 Matrix Inversion

Definition 7.6 Given an $m \times m$ matrix $\mathbf{A}$ if there exists an $m \times m$ matrix $\mathbf{B}$ such that their product is an identity matrix (i.e., $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$), then $\mathbf{B}$ is called the *inverse* of $\mathbf{A}$. The *inverse matrix* is *unique* and usually denoted by $\mathbf{A}^{-1}$. Also, the inverse of $\mathbf{B}$ is $\mathbf{A}$, denoted by $\mathbf{B}^{-1}$.

Note that a square matrix $\mathbf{A}$ can have an inverse if and only if the *row* vectors of $\mathbf{A}$ are linearly independent, or if and only if the *column* vectors of $\mathbf{A}$ are linearly independent. There is a simple condition to check for existence of $\mathbf{A}^{-1}$, called nonsingularity.

7.1.7 Determinant of a Matrix

Definition 7.7 Every $n \times n$ square matrix $\mathbf{A}$ has a real number associated with it called the *determinant* of the matrix, denoted by $\det \mathbf{A}$, which is defined as follows:

$$\det \mathbf{A} = \sum_{i=1}^n a_{i1} A_{i1}$$

where A_{i1} is the cofactor of a_{i1} defined as $(-1)^{i+1}$ times the determinant of the submatrix of $\mathbf{A}$ obtained by deleting the i th row and the first column. The matrix $\mathbf{A}$ is *singular* if $\det \mathbf{A} = 0$ and is *nonsingular* if $\det \mathbf{A} \neq 0$.

To illustrate how to calculate the determinant of a matrix, consider the following example:

$$\begin{aligned}
 \det \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} \\
 &= 2A_{11} + 1A_{21} - 2A_{31} \\
 &= 2(-1)^{1+1} \det \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} + 1(-1)^{2+1} \det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} - 2(-1)^{3+1} \det \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \\
 &= 2(1)2 + 1(1)2 - 2(1)(4) \\
 &= -6
 \end{aligned}$$

Thus, matrix $\mathbf{A}$ is nonsingular.

If a matrix $\mathbf{A}$ is nonsingular, then the inverse $\mathbf{A}^{-1}$ exists and can be calculated by

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{\det \mathbf{A}}$$

where $\text{adj } \mathbf{A}$ is called the *adjoint matrix* of $\mathbf{A}$, which is defined as the transpose of the matrix whose ij element is A_{ij} , the cofactor of a_{ij} . However, this method for finding the inverse of a matrix is not as effective as the method using the elementary row operations to be given in the following section.

Consider the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $n \times n$ nonsingular matrix, $\mathbf{b}$ is an n vector, and $\mathbf{x}$ is an n vector of unknowns. Then according to the *Cramer's rule*, the unique solution to this system is given by

$$x_j = \frac{\det \mathbf{A}_j}{\det \mathbf{A}} \quad \text{for } j = 1, 2, \dots, n$$

where $\mathbf{A}_j$ is obtained from $\mathbf{A}$ by replacing the j th column of $\mathbf{A}$ by $\mathbf{b}$. However, this method is not as effective as that of the elementary row operations.

7.1.8 Upper and Lower Triangular Matrices

Definition 7.8 A square matrix $\mathbf{A} = (a_{ij})$ is called an *upper triangular matrix* if $a_{ij} = 0$ for all $i > j$. Matrix $\mathbf{A}$ is called a *lower triangular matrix* if $a_{ij} = 0$ for all $i < j$. A square matrix $\mathbf{D}$ is called a *diagonal matrix* with diagonal elements $d_{11}, d_{22}, \dots, d_{nn}$ if all other elements ($d_{ij}, i \neq j$) are equal to 0. The diagonal matrix is denoted by $\mathbf{D} = \text{diag}\{d_{11}, d_{22}, \dots, d_{nn}\}$.

The triangular matrix (either upper or lower) with nonzero diagonal elements has an inverse. Note that in solving a system of equations, finding an *equivalent* triangular

coefficient matrix requires about one-half the computational operations as in finding the inverse of a basis matrix. Two systems of linear equations are said to be *equivalent* if their solution sets (including empty set) are the same.

A diagonal matrix $\mathbf{D}$ with diagonal elements $d_{11}, d_{22}, \dots, d_{nn}$ is a special form of a triangular matrix (either upper or lower) in which $d_{ij} = 0$ for all $i \neq j$. Let $\mathbf{D}$ be the diagonal matrix obtained from matrix $\mathbf{A}$ by the elementary row operations, it can be shown that $\det \mathbf{A} = \det \mathbf{D}$ and therefore

$$\det \mathbf{A} = d_{11}, d_{22}, \dots, d_{nn}$$

If any diagonal element $d_{ii} = 0$, then $\det \mathbf{A} = 0$ and hence matrix $\mathbf{A}$ is *singular*.

7.2 USES OF ELEMENTARY ROW OPERATIONS

Given a matrix $\mathbf{A}$ (or a collection of *row vectors*), we can perform some *elementary row operations* (or *row operations*, for short). In the context of this textbook, these operations have the following uses:

- Finding the rank of $\mathbf{A}$ (or the number of linearly independent row vectors)
- Calculating the inverse of $\mathbf{A}$ if exists
- Converting to an upper or a lower triangular matrix
- Converting to a diagonal matrix
- Calculating the determinant of a matrix
- Solving a system of linear equations

A row operation on a matrix is one of the following operations:

1. Interchange of any two rows ($R_i \leftrightarrow R_j$).
2. Multiply any row R_i by a scalar $k \neq 0$ and use the resultant row kR_i to replace R_i .
3. Addition to any row R_j of a nonzero scalar multiple of another row R_i , or use $R_j + kR_i$ to replace R_j .

We can also view a matrix $\mathbf{A}$ as a collection of *column vectors* and can perform some *elementary column operations* (or *column operations*, for short) on a matrix in a way similar to row operations except that rows are replaced by columns.

7.2.1 Finding the Rank of a Matrix

To find the rank of a given matrix, we either apply row operations to find the row rank or apply column operations to find the column rank. The following example is to show an application of row operations.

Example 7.1 Find the rank of $\mathbf{A}$ defined by

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 3 & 1 & 9 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

We apply row operations 2 and 3 to $\mathbf{A}$, and *attempt* to make as many columns as possible to become *distinct unit vectors*. For this example, we attempt to obtain three distinct unit vectors in the following order:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To obtain the first unit column vector, we multiply row R_1 by 0.5 and use $0.5R_1$ to replace R_1 . Then use $R_2 - 1R_1$ to replace R_2 , and use $R_3 - 1R_1$ to replace R_3 . To obtain the second unit column vector, we multiply R_2 by 0.4 and use $0.4R_2$ to replace R_2 . We then use $R_1 - 0.5R_2$ to replace R_1 and use $R_3 - 0.5R_2$ to replace R_3 :

$$\mathbf{A} \rightarrow \begin{pmatrix} 1 & 0.5 & 1 & 1.5 \\ 0 & 2.5 & 0 & 7.5 \\ 0 & 0.5 & 0 & 1.5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

After making two distinct unit columns, we find that the third row becomes a zero vector. This indicates that row 3 is a linear combination of rows 1 and 2. In other words, the maximum number of linearly independent vectors is 2, or the rank of $\mathbf{A}$ is 2.

7.2.2 Calculating the Inverse of a Matrix

An $m \times n$ matrix $\mathbf{A}$ is *invertible* or *nonsingular* if it contains m linearly independent rows. To calculate *the inverse* of a square matrix, we construct an augmented matrix $(\mathbf{A}|\mathbf{I})$ and perform row operations on it until the augmented matrix becomes $(\mathbf{I}|\mathbf{B})$. Then $\mathbf{B} = \mathbf{A}^{-1}$.

Example 7.2 Find the inverse of matrix $\mathbf{A}$ defined by

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Construct the matrix by augmenting an identity matrix of the same size and perform row operations 2 and 3 repeatedly until an identity matrix is obtained on the left-hand part of the augmented matrix.

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 5/2 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/5 & -1/5 & 0 \\ 0 & 1 & 0 & -1/5 & 2/5 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & -1/5 & 1 \\ 0 & 1 & 0 & -1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right)$$

Therefore, we obtain

$$\mathbf{A}^{-1} = \begin{pmatrix} -2/5 & -1/5 & 1 \\ -1/5 & 2/5 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

7.2.3 Converting to a Triangular Matrix

Example 7.3 Find the upper triangular, lower triangular, and diagonal matrices for

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

Apply the elementary row operations on $\mathbf{A}$.

For upper triangular matrix

$$\mathbf{A} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

For lower triangular matrix

$$\mathbf{A} \rightarrow \begin{pmatrix} 2/3 & 0 & 0 \\ -1 & 3 & 0 \\ 2/3 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2/3 & 0 & 0 \\ -1 & 3 & 0 \\ 2/3 & 0 & 1 \end{pmatrix}$$

From upper triangular to diagonal

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From lower triangular to diagonal matrix

$$\begin{pmatrix} 2/3 & 0 & 0 \\ -1 & 3 & 0 \\ 2/3 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7.2.4 Calculating the Determinant of a Matrix

The determinant of a matrix $\mathbf{A}$ can be obtained by calculating the product of the diagonal elements of any one of its equivalent upper triangular, lower triangular, or diagonal matrix. From Example 6.3, we may compute

$$\det \mathbf{A} = 1 \times 2 \times 1 = \frac{2}{3} \times 3 \times 1 = 2$$

7.2.5 Solving a System of Linear Equations

There are two commonly used methods for solving a system of linear equations: (1) matrix inversion and (2) matrix triangularization. The former method, called *Gauss–Jordan reduction*, is well known in introductory operations research texts. The latter method, called *Gaussian reduction*, is more efficient and converts a given augmented matrix to a triangular matrix, either upper or lower triangular.

We shall first use the first method to solve a system of equations. Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is $m \times n$, $\mathbf{x}$ is $n \times 1$, and $\mathbf{b}$ is $m \times 1$. Such a system has one of three cases when solved: (1) a *unique solution*, (2) an *infinite number of solutions*, and (3) *no solution*. To know which case holds for a given system, we must use the concept of rank and the *augmented matrix* of the system denoted $(\mathbf{A}|\mathbf{b})$.

1. The system has a unique solution if $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = n$.
2. The system has an infinite number of solutions if $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) < n$.
3. The system has no solution if $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}|\mathbf{b})$.

Example 7.4 (Unique Solution) Solve the following system:

$$x_1 + x_2 + x_3 = 4$$

$$x_1 + 3x_2 + 3x_3 = 2$$

$$x_1 + x_2 + 2x_3 = 6$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{rank}(\mathbf{A}) = 3$$

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 2 & 6 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Because $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = 3 = n$, the system has a unique solution

$$x_1 = 5, \quad x_2 = -3, \quad x_3 = 2$$

Example 7.5 (Infinite Number of Solutions) Solve the following system:

$$x_1 + x_2 + x_3 + x_4 = 4$$

$$x_1 + 3x_2 + 3x_3 = 2$$

$$x_1 + x_2 + 2x_3 - x_4 = 6$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 0 \\ 1 & 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\Rightarrow \text{rank}(\mathbf{A}) = 3$$

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 0 & 2 \\ 1 & 1 & 2 & -1 & 6 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & -1 & -2 \\ 0 & 0 & 1 & -2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3/2 & 5 \\ 0 & 1 & 0 & -1/2 & -1 \\ 0 & 0 & 1 & -2 & 2 \end{array} \right) \\ \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3/2 & 5 \\ 0 & 1 & 0 & 3/2 & -3 \\ 0 & 0 & 1 & -2 & 2 \end{array} \right)$$

Because $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = 3 < n = 4$, the system has an infinite number of solutions. Note that the final equivalent system of equations reads

$$\begin{aligned}x_1 &= 5 - \frac{3}{2}x_4 \\x_2 &= -3 - \frac{3}{2}x_4 \\x_3 &= 2 + 2x_4\end{aligned}$$

Clearly, the system has an infinite number of solutions because an infinite number of possible values can be assigned to x_4 .

Example 7.6 (No Solution) Consider the following system:

$$\begin{aligned}2x_1 + x_2 + 2x_3 &= 6 \\x_1 + 3x_2 + x_3 &= 9 \\x_1 + x_2 + x_3 &= 3\end{aligned}$$

Applying the row operations to matrix $\mathbf{A}$, we have

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 5/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(\mathbf{A}) = 2$$

$$\begin{aligned}(\mathbf{A}|\mathbf{b}) &= \left(\begin{array}{ccc|c} 2 & 1 & 2 & 6 \\ 1 & 3 & 1 & 9 \\ 1 & 1 & 1 & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1/2 & 1 & 3 \\ 0 & 5/2 & 0 & 6 \\ 0 & 1/2 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 24/5 \\ 0 & 1 & 0 & 12/5 \\ 0 & 0 & 0 & -6/5 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \\ &\Rightarrow \text{rank}(\mathbf{A}|\mathbf{b}) = 3\end{aligned}$$

Because $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}|\mathbf{b})$, the system has no solution. Note that the third row of the last equivalent system gives the equation

$$0x_1 + 0x_2 + 0x_3 = 1$$

Clearly, the left-hand side of this equation never equals to the right-hand side. Hence, the system has no solution.

Now let us use the matrix triangularization method to solve a system of linear equations. Basically, the procedure performs a sequence of row operations on the augmented matrix $(\mathbf{A}|\mathbf{b})$ until a triangular matrix appears on the left-hand side. Then for the lower triangular matrix, use forward substitutions to find the solution; or for the upper triangular matrix, use backward substitutions to find the solution.

Example 7.7 Solve the following system by matrix triangularization

$$\begin{aligned}x_1 + x_2 + x_3 &= 4 \\x_1 + 3x_2 + 3x_3 &= 2 \\x_1 + x_2 + 2x_3 &= 6\end{aligned}$$

Perform row operations on the augmented matrix until an upper triangular matrix is obtained:

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 2 & 6 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

The equivalent system of equations becomes

$$\begin{aligned}x_1 + x_2 + x_3 &= 4 \\2x_2 + 2x_3 &= -2 \\x_3 &= 2\end{aligned}$$

Applying backward substitutions, we have $x_3 = 2 \rightarrow x_2 = -3 \rightarrow x_1 = 5$.

Likewise, we perform row operations on the augmented matrix until a lower triangular matrix is obtained:

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 2 & 6 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2/3 & 0 & 0 & 10/3 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 2 & 6 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2/3 & 0 & 0 & 10/3 \\ -1/2 & 3/2 & 0 & -7 \\ 1 & 1 & 2 & 6 \end{array} \right)$$

Applying forward substitutions, we have $x_1 = 5 \rightarrow x_2 = -3 \rightarrow x_3 = 2$.

7.3 THE DUAL LINEAR PROGRAM

Every linear program, whether expressed in standard form or not, has another linear program associated with it called the “*dual*”. In this context, the *given* or *original* LP problem will be referred to as the “*primal*”. The dual problem complements its *primal* in many ways. In problem formulation, for example, if the primal (P) problem is a maximization, then the dual (D) is a minimization, and vice versa. Moreover, both problems share all the data (parameters) found in $\mathbf{A}$, $\mathbf{c}$, and $\mathbf{b}$.

Knowing the relations between an LP and its dual is vital to understanding advanced topics in linear and nonlinear programming such as economic interpretation, sensitivity analysis, and development of dual and primal–dual simplex methods.

After we explain the formulation of the dual problem from the primal, first in standard form and then in *arbitrary* form, we will provide an economic interpretation of the dual problem using an insurance portfolio problem as an example. Then we will review the duality theory as to the relations between their respective feasible solutions and their respective optimal solutions in this chapter. Other contributions from duality theory will be introduced in later chapters when needed.

7.3.1 The Linear Program in Standard Form

First, let us define the linear programming problem in standard form:

$$\begin{aligned}
 \text{(P)} \quad & \text{maximize} \quad z = \sum_j c_j x_j \\
 & \text{subject to} \quad \sum_j a_{ij} x_j \leq b_i \quad (i = 1, 2, \dots, m) \\
 & \quad \quad \quad x_j \geq 0 \quad (j = 1, 2, \dots, n)
 \end{aligned}$$

or in matrix form,

$$\begin{aligned}
 \text{(P')} \quad & \text{maximize} \quad z = \mathbf{c}^T \mathbf{x} \\
 & \text{subject to} \quad \mathbf{Ax} \leq \mathbf{b} \\
 & \quad \quad \quad \mathbf{x} \geq \mathbf{0}
 \end{aligned}$$

Throughout this text, a linear program is said to be in *standard form* if (1) the objective function is maximized, (2) all the constraints are of $\leq$ form, and (3) all continuous variables are ≥ 0 with no finite upper bound. However, all parameters b_i , c_j , and a_{ij} may be positive, negative, or zero.

Any LP problem that does not conform to conditions (1)–(3) is in *nonstandard form*, which can be converted to standard by simple substitutions. Various non-standard forms are as follows:

- Minimization problem
- Inequality of $\geq$ form
- Equation (equality constraint)
- Unrestricted variable (continuous or integer)
- Variable with a lower bound other than 0
- Variable with a finite upper bound

The conversion procedures for an LP problem are identical to those for an MIP problem described in Section 1.2. For ease of presentation, we shall use the standard LP form for the remainder of the text unless specified otherwise.

The above mathematical definition of an LP problem implies the following assumptions:

- *Divisibility assumption* for each continuous variable ($x_j \geq 0$)
- *Certainty (constant) assumption* for each input parameter (c_j, a_{ij}, b_i)
- *Proportionality assumption* for each term in the constraint and objective function ($a_{ij}x_j, c_jx_j$)
- *Additivity and separability assumption* for each combined function in the objective and constraints ($\sum_j c_jx_j, \sum_j a_{ij}x_j$)
- *Single-objective assumption* ($\max$ or $\min z = \sum_j c_jx_j$)
- *Simultaneousness (conjunction) assumption* for the system of all constraint equations and inequalities ($\sum_j a_{ij}x_j \leq b_i, i = 1, 2, \dots, m$)

Note that these assumptions are the same as those imposed on an MIP problem except for the absence of the integrality assumption. The implications of *all* other assumptions are the same as those described in Section 2.1. We shall not reiterate here.

7.3.2 Formulating the Dual Problem

To formulate the dual of P, we first detach variables from the coefficients and form the following augmented matrix by combining **A**, **b**, and $\mathbf{c}^T$.

$$\begin{array}{ccccccc}
 & & & & & & \text{Dual variable} \\
 & x_1 & x_2 & \dots & x_n & \leq & \\
 \left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c}^T & 1 \end{array} \right) = & \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\ \hline c_1 & c_2 & \dots & c_n & 1 \end{array} \right) & \begin{array}{l} u_1 \geq 0 \\ u_2 \geq 0 \\ \vdots \\ u_m \geq 0 \end{array}
 \end{array}$$

The dual problem is formulated by the following procedure:

1. Assign a *nonnegative dual variable* to each corresponding constraint, denoted by $u_1, u_2, \dots, u_i, \dots, u_m \geq 0$, or $\mathbf{u} \geq \mathbf{0}$.

2. Construct *dual constraints* with respect to variables x_j :

$$a_{1j}u_1 + a_{2j}u_2 + \cdots + a_{mj}u_m = \sum_{i=1}^m a_{ij}u_i \geq c_j, \quad j = 1, 2, \dots, n$$

or, $\mathbf{A}^T \mathbf{u} \geq \mathbf{c}$

3. Construct *dual objective function*

$$\text{minimize } w = b_1u_1 + b_2u_2 + \cdots + b_iu_i + \cdots + b_mu_m = \sum_{i=1}^m b_iu_i$$

or, minimize $w = \mathbf{b}^T \mathbf{u}$

The dual problem is recapped as follows:

$$\begin{aligned} \text{(D)} \quad & \text{Minimize} \quad w = \sum_{i=1}^m b_iu_i \\ & \text{subject to} \quad \sum_{i=1}^m a_{ij}u_i \geq c_j \quad j = 1, 2, \dots, n \\ & \quad \quad \quad u_i \geq 0 \quad i = 1, 2, \dots, m \\ \text{(D')} \quad & \text{Minimize} \quad w = \mathbf{b}^T \mathbf{u} \\ & \text{subject to} \quad \mathbf{A}^T \mathbf{u} \geq \mathbf{c} \\ & \quad \quad \quad \mathbf{u} \geq \mathbf{0} \end{aligned}$$

Basically, the formulation rule state that if a given (primal) problem is a maximization problem with all constraints of $\leq$ form and all variables ≥ 0 , then the dual problem must be a minimization problem with all dual constraints of $\geq$ form and all dual variables ≥ 0 . Both primal and dual problems share the same set of coefficient matrix and vectors of objective coefficients and right-hand side constants, while the dual problem takes on the *transpose* of the original matrix and vectors given in the primal problem. That is, in matrix notation, the dual problem uses $\mathbf{A}^T$, $\mathbf{c}$, and $\mathbf{b}^T$ as the coefficient matrix, right-hand side column, and objective coefficients, respectively. In other words, the transpose of a matrix implies that the objective coefficients of the primal become the right-hand sides of the dual, and vice versa. The number of variables in the primal equals the number of constraints in the dual, and vice versa. Relative to the dual problem, *primal* variables and *primal constraints* are used for referring the variables and constraints defined in the given problem.

Clearly, the dual of the dual is the primal because the transpose of the transpose of a matrix is itself. This implies that we may also refer the minimization problem as the primal and the maximization problem as the dual. Therefore, the format in problem D or D' will also be treated as the *standard minimization problem*.

Example 7.8 If the primal problem is

Maximize $z = x_1 + 2x_2 - 8x_3$
subject to $x_1 + 3x_2 + 5x_3 \leq 8$
 $2x_1 - 5x_3 \leq 7$
 $x_1, x_2, x_3 \geq 0$

then the dual problem is

Minimize $w = 8u_1 + 7u_2$
subject to $u_1 + 2u_2 \geq 1$
 $3u_1 \geq 2$
 $5u_1 - 5u_2 \geq -8$
 $u_1, u_2 \geq 0$

What if the given problem is not in standard form? Straightforwardly, we may first convert the given problem to a standard problem as usual, then formulate the dual from the converted standard problem, and finally convert this dual problem back to the original format. It can be shown that the result of this three-step procedure corresponds to the primal–dual formulation rules listed in Table 7.1. The reader is encouraged to verify these rules in exercises by the three-step procedure.

Example 7.9 Formulate the dual problem of the following LP:

Maximize $z = 2x_1 - x_2 + 5x_3 + 3x_4$
subject to $x_1 + 2x_2 + 3x_3 - x_4 \geq 5$
 $2x_1 - 3x_2 + x_3 + 2x_4 \leq 12$
 $x_2 - x_3 + x_4 = -3$
 $x_1, x_2, x_4 \geq 0, x_3 \leq 0$

TABLE 7.1 Correspondence of Primal–Dual Formulation

Maximization Problem	Minimization Problem
Constraint i	Variable I
$\leq$	≥ 0
$\geq$	≤ 0
$=$	Unrestricted in sign
Variable j	Constraint j
≥ 0	$\geq$
≤ 0	$\leq$
Unrestricted in sign	$=$
Objective row	Right-hand side column
Right-hand side column	Objective row

The dual problem is

$$\begin{aligned}
 &\text{Minimize} && w = 5u_1 + 12u_2 - 3u_3 \\
 &\text{subject to} && u_1 + 2u_2 \geq 2 \\
 &&& 2u_1 - 3u_2 + u_3 \leq -1 \\
 &&& 3u_1 + u_2 - u_3 \leq 5 \\
 &&& -u_1 + 2u_2 + u_3 \geq 3 \\
 &&& u_1 \leq 0, u_2 \geq 0, u_3 \text{ unrestricted in sign}
 \end{aligned}$$

7.3.3 Economic Interpretation of the Dual

Suppose the primal problem represents a resource allocation problem at a manufacturing plant; that is, x_j is the quantity of product j to produce in a given period (say one month) and the limited availability of raw material i is represented by $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$, for each $i = 1, \dots, m$. The plant manager wants to maximize profit in the particular production mix $(x_1, x_2, \dots, x_n)$ found to represent optimal utilization of available raw materials $b_1, b_2, \dots, b_m$.

Now, consider the company's risk manager, who reports to the VP of finance. He is interested in insuring the raw materials on hand against loss, but only wants to insure them (place a per unit valuation on each) up to their value in producing products that result in sales and profit (represented by $c_1, c_2, \dots, c_n$ for each potential product manufactured).

If we let u_i be the per unit insured value of raw material i , the total valuation of resource i on-hand would be $b_i u_i$. We shall assume that $u_i \geq 0$, and otherwise no cost to dispose of unused resource i . The objective function for minimizing the cost of insurance is to minimize $w = b_1 u_1 + b_2 u_2 + \cdots + b_m u_m$.

Furthermore, the combined values of the various raw materials used to make say one unit of product j must be at least c_j , the profit from producing one unit of product j ; in mathematical notation: $a_{1j}u_1 + a_{2j}u_2 + \cdots + a_{mj}u_m \geq c_j$.

Because the above relationship must hold for each product $j = 1, \dots, n$, the risk manager's linear program to choose the least cost insurance portfolio is

$$\begin{aligned}
 &\text{Minimize} && \mathbf{b}^T \mathbf{u} \\
 &\text{subject to} && \mathbf{A}^T \mathbf{u} \geq \mathbf{c} \\
 &\text{and} && \mathbf{u} \geq \mathbf{0}
 \end{aligned}$$

which is recognized as the dual problem to the primal production mix problem.

7.3.4 Importance of the Dual

We have just seen that the duality relationship for the standard form LP explained by the total insurance valuation of on-hand raw materials (**b**) should be precisely equal to the maximum dollar value that can be extracted from them, using the company's current technology (**A**) and profit per unit for each of n products (**c**). There are many more such insights to be gained from the dual LP, regardless of whether the primal represents the classic resource allocation model or some other model. Duality plays a central role in linear programming. The primal–dual relationships presented here and in the following section help establish the simplex method, and then develop an alternative version known as the dual simplex algorithm. It turns out that whether one is solving the primal or dual LP via the simplex (or related) algorithm, one is automatically solving the dual LP as well. More theory is needed to justify this statement, presented next.

7.4 RELATIONSHIPS BETWEEN PRIMAL AND DUAL SOLUTIONS

There are a series of primal–dual relationships the reader needs to know. Let $\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_n^0)$ represent any feasible solution to the maximizing (primal) problem, with objective value $z^0 = \sum_{j=1}^n c_j x_j^0 = \mathbf{c}^T \mathbf{x}^0$. Let $\mathbf{u}^0 = (u_1^0, u_2^0, \dots, u_m^0)$ be feasible for the minimizing (dual) problem, with objective function value $w^0 = \sum_{i=1}^m b_i u_i^0 = \mathbf{b}^T \mathbf{u}^0$.

7.4.1 Relationships Between All Primal and All Dual Feasible Solutions

Since the primal problem can be either maximization or minimization and the dual problem can be either maximization or minimization too, we shall simply use the maximization problem or minimization problem in this section to ease the presentation of primal–dual relations.

The Weak Duality Theorem If $\mathbf{x}^0$ is feasible a maximization problem, and $\mathbf{u}^0$ is feasible for the associated minimization problem, then the objective value of the maximization problem is a lower bound of the objective value of the associated minimization problem, or mathematically $z^0 = \sum_{j=1}^n c_j x_j^0 \leq w^0 = \sum_{i=1}^m b_i u_i^0$.

Proof Since $\mathbf{x}^0$ is feasible to the maximizing problem, we have

$$z^0 = \sum_{j=1}^n c_j x_j^0 \quad (7.1)$$

$$\text{subject to } \sum_{j=1}^n a_{ij} x_j^0 \leq b_i \quad (i = 1, 2, \dots, m) \quad (7.2)$$

$$x_j^0 \geq 0 \quad (j = 1, 2, \dots, n) \quad (7.3)$$

Since $\mathbf{u}^0$ is feasible to the minimizing problem, we have

$$w^0 = \sum_{i=1}^m b_i u_i^0 \quad (7.1')$$

$$\text{subject to } \sum_{i=1}^m a_{ij} u_i^0 \geq c_j \quad (j = 1, 2, \dots, n) \quad (7.2')$$

$$u_i^0 \geq 0 \quad (i = 1, 2, \dots, m) \quad (7.3')$$

Premultiplying (7.2) by (7.3') and summing over i , we obtain

$$\sum_{i=1}^m u_i^0 \sum_{j=1}^n a_{ij} x_j^0 \leq \sum_{i=1}^m b_i u_i^0 \quad (7.4)$$

Premultiplying (7.2') by (7.3) and summing over j , we obtain

$$\sum_{j=1}^n x_j^0 \sum_{i=1}^m a_{ij} u_i^0 \geq \sum_{j=1}^n c_j x_j^0 \quad (7.4')$$

Rearranging and combining (7.4) and (7.4'), we get

$$\sum_{j=1}^n c_j x_j^0 \leq \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_j^0 u_i^0 \leq \sum_{i=1}^m b_i u_i^0 \quad (7.5)$$

or, by definition, $z^0 \leq w^0$. Expressing (7.5) in matrix notation, we have

$$\mathbf{c}^T \mathbf{x}^0 \leq (\mathbf{u}^0)^T \mathbf{A} \mathbf{x}^0 \leq \mathbf{b}^T \mathbf{u}^0 \quad (7.6)$$

■

7.4.2 Relationship Between Primal and Dual Optimum Solutions

The Duality Theorem There are three possible relationships between the primal and dual problems:

1. If one problem has a feasible solution $\mathbf{x}^*$ with a bounded objective value z^* , then the other problem has a feasible solution $\mathbf{u}^*$ with a bounded objective value w^* . Furthermore, both problems have a finite optimum solution with the relation $\mathbf{c}^T \mathbf{x}^* = (\mathbf{u}^*)^T \mathbf{A} \mathbf{x}^* = \mathbf{b}^T \mathbf{u}^*$.

2. If one problem has a feasible solution with an unbounded objective value, then the other problem has no feasible solution.
3. If one problem has no feasible solution, then the other problem has either no feasible solution or an unbounded solution.

Example 7.10 The reader should verify graphically that both primal and dual problems below are infeasible.

$$\begin{array}{ll}
 \text{(P)} \quad \text{maximize} & z = x_1 + 2x_2 \\
 \text{subject to} & x_1 - x_2 \leq 0 \\
 & -x_1 + x_2 \leq -1 \\
 & x_1, x_2 \geq 0
 \end{array}
 \quad
 \begin{array}{ll}
 \text{(D)} \quad \text{minimize} & w = -u_2 \\
 \text{subject to} & u_1 - u_2 \geq 1 \\
 & -u_1 + u_2 \geq 2 \\
 & u_1, u_2 \geq 0
 \end{array}$$

7.4.3 Relationships Between Each Complementary Pair of Variables at Optimum

We now convert problem P to *equality* constraints by *adding* slack variables x_{s_i} ($i = 1, 2, \dots, m$).

$$\begin{array}{ll}
 \text{Maximize} & z = \sum_j c_j x_j \\
 \text{subject to} & \sum_j a_{ij} x_j + x_{s_i} = b_i \quad (i = 1, 2, \dots, m) \\
 & x_j \geq 0 \quad (j = 1, 2, \dots, n)
 \end{array}$$

Likewise, convert problem D to dual equality constraints by *subtracting* surplus variables u_{s_j} ($j = 1, 2, \dots, n$).

$$\begin{array}{ll}
 \text{Minimize} & w = \sum_{i=1}^m b_i u_i \\
 \text{subject to} & \sum_{i=1}^m a_{ij} u_i - u_{s_j} = c_j \quad j = 1, 2, \dots, n \\
 & u_i \geq 0 \quad j = 1, 2, \dots, m
 \end{array}$$

Complementary Slackness Theorem Let $x_1^*, x_2^*, \dots, x_n^*$ be an optimum solution to the primal problem P and $x_{s_1}^*, x_{s_2}^*, \dots, x_{s_m}^*$ be the associated slack variables. Also, let $u_1^*, u_2^*, \dots, u_m^*$ be an optimum solution to the dual problem D and $u_{s_1}^*, u_{s_2}^*, \dots, u_{s_n}^*$

be the associated surplus variables. Then, the following relation holds for each of the complementary pairs of variables:

$$\begin{aligned}x_{s_i}^* u_i^* &= 0 \quad \text{for } i = 1, 2, \dots, m \\ u_{s_j}^* x_j^* &= 0 \quad \text{for } j = 1, 2, \dots, n\end{aligned}$$

Proof From the duality theorem, we have

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_j^* u_i^* = \sum_{i=1}^m b_i u_i^*$$

Subtracting the middle term from the rightmost term, we have

$$\sum_{i=1}^m \left(b_i - \sum_{j=1}^n a_{ij} x_j^* \right) u_i^* = 0$$

or

$$\sum_{i=1}^m x_{s_i}^* u_i^* = 0$$

But because $u_i^* \geq 0$ for all i , this equality implies that $x_{s_i}^* u_i^* = 0$ for $i = 1, 2, \dots, m$.

Likewise, subtracting the leftmost term from the middle term, we have

$$\sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} u_i^* - c_j \right) x_j^* = 0$$

or

$$\sum_{j=1}^n u_{s_j}^* x_j^* = 0$$

But because $x_j^* \geq 0$ for all j , this equality implies $u_{s_j}^* x_j^* = 0$ for $j = 1, 2, \dots, n$, completing the proof. ■

Example 7.11 Consider the following LP problem (P) and its dual (D) (with slack variables and surplus variables).

$$\begin{aligned}(\text{P}) \quad & \text{maximize} \quad z = x_1 - 3x_2 + x_3 \\ & \text{subject to} \quad -x_1 + 2x_2 + x_3 + x_{s_1} = 10 \\ & \quad \quad \quad 3x_1 - 2x_2 + 3x_3 + x_{s_2} = 20 \\ & \quad \quad \quad \mathbf{x} \geq \mathbf{0}\end{aligned}$$

$$\begin{aligned}(\text{D}) \quad & \text{minimize} \quad w = 10u_1 + 20u_2 \\ & \text{subject to} \quad -u_1 + 3u_2 - u_{s_1} = 1 \\ & \quad \quad \quad 2u_1 - 2u_2 - u_{s_2} = -3 \\ & \quad \quad \quad u_1 + 3u_2 - u_{s_3} = 1 \\ & \quad \quad \quad \mathbf{u} \geq \mathbf{0}\end{aligned}$$

Suppose we know that the optimal solution to D is $u_1 = 0$, $u_2 = 1/3$, $w^* = 20/3$, which implies that $u_{s_1} = 0$, $u_{s_2} = -7/3$, $u_{s_3} = 0$. By the duality and the complementary theorems, we can obtain that $z^* = w^* = 20/3$, $x_{s_2} = 0$, and $x_2 = 0$.

Let us look at one more example.

Example 7.12 Consider the following LP problem (P) and its dual (D).

$$\begin{array}{ll}
 \text{(P) maximize} & z = 7x_1 + 11x_2 \\
 \text{subject to} & x_1 + x_2 + x_{s_1} = 11 \\
 & 2x_1 - x_2 + x_{s_2} = 5 \\
 & -3x_1 - 2x_2 + x_{s_3} = -20 \\
 & \mathbf{x} \geq \mathbf{0}
 \end{array}
 \qquad
 \begin{array}{ll}
 \text{(D) minimize} & w = 11u_1 + 5u_2 - 20u_3 \\
 \text{subject to} & u_1 + 2u_2 - 3u_3 - u_{s_1} = 7 \\
 & u_1 - u_2 - 2u_3 - u_{s_2} = 11 \\
 & \mathbf{u} \geq \mathbf{0}
 \end{array}$$

Given that the optimal solution to P is $x_1 = 0$, $x_2 = 11$, $z^* = 121$, which implies that $x_{s_1} = 0$, $x_{s_2} = 16$, and $x_{s_3} = 2$. By strong duality and the complementary rule, we can obtain that $w^* = z^* = 121$, $u_2 = 0$, $u_3 = 0$, and $u_{s_2} = 0$, which implies $u_1 = 11$ and $u_{s_1} = 4$. Hence, all information about the dual optimal solution is also obtained.

7.5 NOTES

Section 7.2

In some discussions of linear programming, the converted LP with each constraint expressed as an equality is called “standard form,” and the version with all $\leq$ constraints (maximizing objective) is called “canonical form.”

Section 7.4

Duality theory for linear programming was a major focus of applied mathematicians in the early 1950s. The main duality theorem was originally stated by John von Neumann, and proof first appeared in an article by Goldman and Tucker (1956). The interested reader should review the papers by Farkas (1902), the paper by Gale et al. (1951), and then the paper by Goldman and Tucker (1956).

There is a much broader duality theory for convex programming. The classic reference is Rockafellar (1970). For a brief introduction to duality for integer programming, see Section 2.5 of Wolsey (1998). For a comprehensive survey of duality theory and its relation to the concept of relaxation in integer programming, see Chapter II.3 in Nemhauser and Wolsey (1988).

7.6 EXERCISES

7.1 Under what conditions the following expressions are (a) affine combinations and (b) convex combinations?

(1) $\lambda^2 \mathbf{x}^1 + 2\lambda \mathbf{x}^2 - \lambda \mathbf{x}^3 + 3\lambda^2 \mathbf{x}^4$

(2) $2\lambda^3 \mathbf{x}^1 + 3\lambda \mathbf{x}^2 + \lambda^2 \mathbf{x}^3 + 2\lambda \mathbf{x}^4$

7.2 Calculate the determinant of each of the following matrices using the two methods introduced in this chapter.

(1) $\mathbf{A} = \begin{bmatrix} 0 & 3 & 2 & 1 \\ 2 & -1 & 6 & 4 \\ 1 & 4 & -1 & 3 \\ 5 & 2 & 3 & 0 \end{bmatrix}$

(2) $\mathbf{B} = \begin{bmatrix} -3 & 0 & 2 \\ 4 & 7 & 3 \\ 2 & 1 & -5 \end{bmatrix}$

7.3 Generate upper and lower triangular matrices for each of the matrix below. Show your steps.

(1) $\mathbf{A} = \begin{bmatrix} 6 & 9 & 5 \\ -3 & 7 & 4 \\ 4 & 0 & -3 \\ 8 & -1 & 7 \end{bmatrix}$

(2) $\mathbf{B} = \begin{bmatrix} 2 & 5 & -3 & 0 \\ 0 & 4 & 6 & 5 \\ -1 & 2 & 5 & 3 \\ 3 & 3 & 7 & 4 \end{bmatrix}$

7.4 Apply elementary row operations to the matrices in Exercise 7.3 to generate new matrices. Show your steps.

7.5 Prove the following two statements:

(1) If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}(n \times n)$.

(2) If $\mathbf{A}$ is nonsingular, then $\mathbf{AB} = \mathbf{AC}$ implies $\mathbf{B} = \mathbf{C}$.

7.6 Determine the rank of each of the matrices in Exercises 7.2 and 7.3.

- 7.7** Determine whether the following linear system is feasible or not by applying the upper or lower triangular matrix method.

$$2x_1 + x_2 - 3x_3 + x_4 = 9$$

$$-x_1 + 3x_2 + x_3 + 2x_4 = 11$$

$$x_1 - x_2 + 4x_3 = 7$$

$$2x_2 + x_3 - 2x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- 7.8** Determine if the following linear system has (a) no solution, (b) unique solution, or (c) multiple solutions.

$$7x_1 - 3x_2 + x_3 + 4x_4 = 23$$

$$3x_1 + 2x_2 - 4x_3 - x_4 = 19$$

$$x_1 + x_2 + x_3 + 3x_4 = 15$$

$$-x_1 + 3x_2 + 8x_3 + 2x_4 = 29$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- 7.9** Determine the feasibility of the following LP system without solving it. Use at least two methods in this chapter.

$$-x_1 + 3x_2 - 2x_3 = 7$$

$$2x_1 + x_2 - x_3 = 6$$

$$x_1, x_2, x_3 \geq 0$$

- 7.10** Test if the solution $(125/92, 4/23, 91/92)$ is optimal to the following LP. Why or why not?

$$\text{Maximize } z = 2x_1 - 3x_2 + 10x_3$$

$$\text{subject to } -3x_1 + x_2 + 9x_3 \leq 5$$

$$x_1 - 2x_2 + x_3 \leq 2$$

$$6x_1 + 5x_2 + 2x_3 \leq 11$$

$$\mathbf{x} \geq \mathbf{0}$$

7.11 Formulate the dual of the following LP problem.

$$\begin{aligned}
 &\text{Maximize} && z = 11x_1 - 13x_2 + 7x_3 + 9x_4 \\
 &\text{subject to} && -2x_1 + x_2 + 4x_3 - 5x_4 \leq -5 \\
 &&& 5x_1 + 4x_2 - x_3 \leq 17 \\
 &&& 2x_1 + x_3 - x_4 \leq 5 \\
 &&& \mathbf{x} \geq 0
 \end{aligned}$$

7.12 Formulate the dual of the following LP problem.

$$\begin{aligned}
 &\text{Minimize} && z = 11x_1 - 13x_2 + 7x_3 - 9x_4 \\
 &\text{subject to} && 2x_1 - x_2 + 4x_3 - 5x_4 \leq 5 \\
 &&& 5x_1 + 4x_2 - x_3 \leq 17 \\
 &&& -2x_1 + x_3 - x_4 = 5 \\
 &&& x_1, x_2 \geq 0, x_3 \leq 0, x_4 \text{ unrestricted in sign}
 \end{aligned}$$

7.13 Consider the following LP problem and its unique optimal solution. Formulate its dual and figure out as much information about the dual optimal solution from the information given about the primal.

$$\begin{aligned}
 &\text{Maximize} && z = 4x_1 + 3x_2 + x_3 + 7x_4 + 6x_5 \\
 &\text{subject to} && x_1 + 2x_2 + 3x_3 + x_4 - 3x_5 \leq 9 \\
 &&& 2x_1 - x_2 + 2x_3 + 2x_4 + x_5 \leq 10 \\
 &&& -3x_1 + 2x_2 + x_3 - x_4 + 2x_5 \leq 11 \\
 &&& \mathbf{x} \geq 0 \\
 &&& \text{optimal solution is } (7, 10, 0, 0, 6) \text{ with } z^* = 94
 \end{aligned}$$

7.14 Consider the following LP problem and its dual. Given the optimal solution to the dual, figure out as much information about the optimal solution to the primal.

Primal	Dual
maximize $z = 25x_1 - 2x_2 + 16x_3$	minimize $w = 5u_1 + 2u_2 + u_3 + 4u_4 + 3u_5$
subject to $3x_1 + x_2 + 9x_3 \leq 5$	subject to $3u_1 + 5u_2 - 6u_3 + 2u_4 + 2u_5 \geq 25$
$5x_1 + 2x_2 - 4x_3 \leq 2$	$u_1 + 2u_2 + 3u_3 - 7u_4 + 3u_5 \geq -2$
$-6x_1 + 3x_2 + 2x_3 \leq 1$	$9u_1 - 4u_2 + 2u_3 + 5u_4 - u_5 \geq 16$
$2x_1 - 7x_2 + -5x_3 \leq 4$	$\mathbf{u} \geq 0$
$2x_1 + 3x_2 - x_3 \leq 3$	optimal solution :
$\mathbf{x} \geq 0$	$(60/19, 59/19, 0, 0, 0)$ with $w^* = 22$

- 7.15** Consider the following LP problem. Formulate its dual. Given that the optimal solution to the primal problem is $(3, 4, 0, 3)$ with $z^* = 22$, is the optimal solution to the dual degenerate? Why or why not?

$$\begin{array}{ll}\text{Maximize} & z = 3x_1 + x_2 + 2x_3 + 3x_4 \\ \text{subject to} & -x_1 + 3x_2 + x_3 - 2x_4 \leq 17 \\ & 7x_1 + 3x_3 + x_4 \leq 23 \\ & x_1 + 2x_2 \leq 11 \\ & x_2 + 3x_4 \leq 13 \\ & x_1, x_4 \geq 0, x_2 \text{ unrestricted in sign}, x_3 \leq 0\end{array}$$